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ABSTRACT

The first volume of the proceedings of the 21st annual meeting of the International Group for the Psychology of Mathematics Education contains the following 13 full papers: (1) "Some Psychological Issues in the Assessment of Mathematical Performance" (O. Bjorkqvist); (2) "Neuromagnetic Approach in Cognitive Neuroscience" (S. Levanen); (3) "Dilemmas in the Professional Education of Mathematics Teachers" (J. Mousley and P. Sullivan); (4) "Open Toolsets: New Ends and New Means in Learning Mathematics and Science with Computers" (A. A. diSessa); (5) "From Intuition to Inhibition--Mathematics, Education and Other Endangered Species" (S. Vinner); (6) "Distributed Cognition, Technology and Change: Themes for the Plenary Panel" (K. Crawford); (7) "Roles for Teachers, and Computers" (J. Ainley); (8) "Some Questions on Mathematical Learning Environments" (N. Balacheff); (9) "Deepening the Impact of Technology Beyond Assistance with Traditional Formalisms in Order To Democratize Access To Ideas Underlying Calculus" (J. J. Kaput and J. Roschelle); (10) "The Nature of the Object as an Integral Component of Numerical Processes" (E. Gray, D. Pitta, and D. Tall); (11) "Unifying Cognitive and Sociocultural Aspects in Research on Learning the Function Concept" (R. Hershkowitz and B. B. Schwarz); (12) "Emergence of New Schemes for Solving Algebra Word Problems: The Impact of Technology and the Function Approach" (M. Yerushalmy); (13) "Approaching Geometry Theorems in Contexts: From History and Epistemology To Cognition" (M. A. Mariotti, M. G. Bartolini Bussi, P. Boero, F. Ferri, and R. Garuti). In addition, volume 1 contains three brief reaction papers (two reacting to Gray and one to Mariotti), and 92 one-page abstracts of working group sessions, discussion group sessions, short oral communication sessions, and poster sessions. (ASK)

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# Proceedings

of the

## 21st Conference

of the

## International Group for the Psychology of Mathematics Education

edited by
Erkki Pehkonen

PME 21
July 14-19, 1997 Lahti, Finland

University of Helsinki
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Volume 1
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*Proceedings of the 21$^{st}$ Conference of the International Group for the Psychology of Mathematics Education*
Volume 1

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# Preface

The PME group has now its 21st conference, and nowadays 20 years is an honourable long time without bigger changes. But during the recent conferences, there has been a vivid discussion on the role of P in the acronym PME. Therefore, I have purposefully selected for the proceedings the cover picture of three circles. During the conference, everybody may ponder what is actually our field of research as the PME group. Is it per definition the intersection $P \cap M \cap E$ of the sets P, M and E? Or is it their union $P \cup M \cup E$ ? Or perhaps some other subset of the union than the intersection? Or is it a larger set which contains the union of P, M and E?

The proceedings has been edited along the models of the earlier ones, especially the Valencia proceedings. All the presentations are in alphabetic order according to the presenting author (underlined). In the introduction part, there is also an index of research reports and short oral communications ordered into the categories given by the authors as their first choice.

During the editing process, I received help from many persons. Especially, my thanks are due to Ms. Marja-Liisa Neuvonen-Rauhala and Ms. Seija Lehtinen (both Lahti Research and Training Centre) for the huge amount of work they have done for this conference and its proceedings.

Erkki Pehkonen

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# INTRODUCTION

# THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

## History and Aims of PME

PME came to existence in 1976 at the third International Congress on Mathematics Education (ICME-3), held in Karlsruhe, Germany. Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gérard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), and Carolyn Kieran (Canada).

The major goals of the Group are: To promote International contacts and the exchange of scientific information in the Psychology of Mathematics Education. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers. And to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

## PME Membership

PME membership is open to people involved in active research consistent with the aims of the Group, or professionally interested in the results of such research. Membership is on annual basis and depends on payment of the membership fee for the current year (January to December). For participants on the PME 20 conference, the membership fee for 1996 is included in the registration fee. Others are requested to write either to their Regional Contact, or directly to the PME Executive Secretary.

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## The Review Process of PME 21

Four research forum proposals were received. Each proposal was reviewed by three PME members with well-known expertise in the topic of the research forum to which the proposal has been submitted. All four proposals were accepted on the grounds of these reviews.

The Program Committee received also 171 research report proposals encompassing a wide variety of topics and approaches. Each proposal was submitted to three PME members with expertise in the specific research domain. The proposals with two or more acceptances were, as a rule, accepted by the Program Committee. The reviews of the rejected papers were looked through by the Program Committee, in order to check whether there were rejecting reviews with poor information. In such cases, one or more members of the Program Committee read the proposal. Altogether 122 research report proposals were accepted.

In addition, 78 short oral proposals and 20 poster proposals were received. Each proposal was read by two members of the Program Committee (three when a proposal was rejected). On the basis of these reviews, 58 short oral proposals and all poster proposals were accepted.

All written comments from the reviewers were forwarded to the authors along with the decision of the Program Committee.

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| PME 19 | 1995 | Recife, Brazil | SE 057 135 | PME Executive Secr. |
| PME 20 | 1996 | Valencia, Spain | | PME Executive Secr. |
| PME-NA 3 | 1981 | Minnesota | ED 223 449 | |
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# PLENARY ADDRESSES

# SOME PSYCHOLOGICAL ISSUES IN THE ASSESSMENT OF MATHEMATICAL PERFORMANCE

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***Abstract.*** *The importance of explicit reference to social psychology in mathematics education is motivated by the problems connected with the introduction of new methods of assessment. The issues of assessment as an agent of change, dynamic assessment, and validity of assessment are analysed from a social constructivist perspective. Teacher concepts of the purposes and modes of assessment are seen as central in the social negotiation of what constitutes good school mathematics.*

## Introduction

The interest in development of assessment procedures in mathematics education has been on a steady increase during the last few years. The variety of innovations covers a wide scale, and some of it would not have been conceived as part of mathematics education just ten years ago. It would be difficult to give a complete overview of the various methods in use today to assess the performance of students in mathematics. It is, however, possible to categorise assessment methods according to their particular characteristics. Webb (1993, pp. 3-4) lists five features that are common to any form of assessment, and which can be used as a framework for such an analysis.

They are 1) a situation, task or question, 2) a response, 3) interpretation of the student´s response by a teacher, 4) assigning some meaning to the interpretation of the student´s responses, and 5) reporting and recording the findings from the assessment.

Niss (1993) gives an even more elaborate structural analysis of assessments in mathematics education according to their *purposes* and *modes*. According to Niss, the purposes fall into three different but not independent categories: the provision of information, the establishing of bases for decisions or actions, and the shaping of social reality (Niss, 1993, p. 6). A given assessment mode may be thought of as a "vector" with a set of components, all or some of which need to be characterised to describe the assessment mode. These components include the subject, the object, the items, the occasions, the procedures and circumstances, the judging and recording, and the reporting (Niss, 1993, pp. 12-14).

Any such characterisation may serve as a basis for further analyses of issues related to assessment in mathematics education. It is my intention to focus on some psychological issues in the assessment of mathematical performance – and a cursory inspection clearly shows that there exist such issues connected to each of the above features, purposes or modes. Those issues that relate to new methods of assessment are evidently of special interest to current research in the psychology of mathematics education.

## Quality in mathematics education

One of the main reasons for the introduction of new methods of assessments has been a clarified view of school mathematics. It is considered important to know what characterises good mathematics and to direct assessment towards objects (tasks and questions) and procedures that reflect such views as closely as possible. Today mathematics is seen as "a complex interlocking system of propositional knowledge, concepts, problems, techniques, methods, working practices, communities, traditions and beliefs" (Pimm, 1996, p. 37). Notions of quality or merit are implicit in all those aspects, but, being largely subjective, are not easy to formulate as goals for mathematics education, nor as objects for assessment unless one is very clear about the specific purpose of the assessment of mathematical performance. A couple of examples will show how psychology enters such discussions of what constitutes good mathematics.

The quality criteria of everyday mathematics, such as for instance used by street vendors, may be expressed in pragmatic terms. The effectiveness of mathematical reasoning is then closely related to the *mental representations* being used (Schliemann, 1995, p. 47). The latter may be equally true if one aims at more abstract mathematical knowledge, but the "best" representations are probably not the same any more.

> The analysis of the strengths and limitations of everyday mathematics leads to the question of how we can design better opportunities for children to develop mathematical knowledge that is wider than they would develop outside of schools, but that preserves the focus on meaning found in everyday situations.
>
> [Schliemann, 1995, p. 52]

In this respect, the challenge for assessment is to provide more information about the mental representations actually used and to monitor their development towards a higher degree of abstraction without loss of meaning.

Problem solving, if identified as a separate field of mathematics education, has quality critera of its own. Many of them relate to mathematics as an efficient tool for general problem solving. Some, however, are quite different and point at problem solving as a higher level mathematical activity, which not everyone is capable of. If the purpose of assessment is to identify individuals with a promising mathematical career, problem solving tasks may well be used. According to Silver (1993, p. 69) *problem posing*, specifically, has long been viewed as a characteristic of creative activity or exceptional talent. The development of assessment tasks that involve problem posing is a nice challenge for mathematics education, and it clearly has interesting psychological aspects.

There are other kinds of tasks that similarly involve students in decision-making with respect to the method of solution, and thus require students to be active in a non-traditional way. Examples of PME research in this direction include the studies of "good questions" (a particular form of open-ended mathematics tasks) by Clarke & Sullivan (1992) and of the properties of "powerful tasks" (Krainer, 1992).

If mathematical power is identified with the capacity to solve non-routine problems, open-ended tasks are an appropriate vehicle for instruction and assessment of students' learning (Clarke & Sullivan, 1992, p. 143). But the assessment of students' performance on open-ended tasks is by no means a routine thing, and the identification of what quality mathematics is does not immediately give quality asssessment.

## The social constructivist perspective

It is important to put the characterisation of "good mathematics" into a larger perspective. I will choose social constructivism, as applied to mathematics education (Ernest, 1991), and focus on the interaction between individual knowledge and collective knowledge which is central to any variety of social constructivism. It is assumed that collective knowledge is built up from contributions by individuals, and that the construction of individual knowledge is influenced by collective knowledge (Björkqvist, 1993).

The interdependence can be expressed in terms of survival. Collective knowledge is for human survival – and collective knowledge itself survives only if it is available again and again for new individuals as they construct their personal knowledge.

A conclusion to be drawn from this is that, in assessment, *viability of knowledge is a criterion for quality*. This is true for individual knowledge as well as collective knowledge (Björkqvist, 1995).

The goals of education, including mathematics education, are derived from survival of the individual and society. It means considering, e.g., the role of mathematics for science and technology, the construction of new mathematics in society, mathematics and democracy, gender and mathematics, etc. Mathematics education, just like mathematics itself, should be *dynamic* and respond to changes in society.

From an analytical point of view, this is almost self-evident. But, on the other hand, the picture of mathematics in society is not necessarily a consistent one. Mathematics is very often seen as a static part of human knowledge – a trusty pillar to lean on in a changing world. Assessment in mathematics has had its emphasis on those stable aspects, and the transformation in the direction of more dynamic methods has been slow.

In particular, there is an increasing need to assess the dynamic aspects of the personal mathematics of the students. This includes assessing the processes by which students create knowledge that is new to them and assessing the quality of their constructs with respect to viability in foreseeable and unforeseeable future situations.

Assessment of the relatively static aspects of the mathematics of the students should not disappear. However, from a social constructivist perspective, the aim is not to assess reproductive ability, but rather to promote communicability and interaction between individual and collective mathematical knowledge.

Yackel, Cobb and Wood (1992, p. 69) point out that the goals for students' mathematics education that are valued from a "socioconstructivist" perspective are not linked to performance on specific tasks.

> Instead, they concern the nature of the activity the children engage in as they solve their mathematical problems. In global terms, these goals include that children make meaningful, productive and increasingly powerful mathematical constructions and that they develop the ability to engage in mathematical explanation and argumentation.
>
> [Yackel, Cobb & Wood, 1992, p. 70]

The social construction of classroom knowledge (Jaworski, 1994) exemplifies that "collective" knowledge may refer to different sizes of groups of humans. A classroom of young children is perhaps the best place to study the interaction between individual and collective knowledge, and the results of such research are immediately applicable if they become known by teachers. So far, they have had only a limited impact on the assessment procedures, which, regardless of new methods of teaching being employed, still focus on individual knowledge.

In terms of the features or purposes and modes of assessment, the influence of social constructivism has been to question the relatively few different kinds of assessment procedures actually used by teachers. Assessment is visualised to take place in a variety of settings and under a variety of circumstances. And today – even though we don't exactly know how to make it happen in practice – the necessity of always seeing the individual in a social context has filtered deep into the souls of mathematics educators.

> Mathematics learning in its educational context cannot be fully interpreted intrapersonally because of its social setting. Equally, interpersonal constructs will be inadequate alone since it is always the learner who must make sense and meaning in mathematics. Therefore, it is crucially important to recognize this intra/interpersonal complementarity, to explore its significance and how it influences the quality of mathematical meaning obtained by students in the classroom.
>
> [Balacheff, 1990, p. 139]

## Assessment as an agent of change

Besides having a traditional relation to the goals of mathematics education, assessment has a steering role that can be both unintentional and intentional. This is true at the individual level and at the curriculum level as well. Consciousness of the modes of assessment guides learning, and in the case of large-scale assessments, it also has significant effects on teaching.

Taking advantage of this effect, assessment can be used as a mechanism for monitoring *and* intervening in the educational process. It then has implications for the life chances of the individual and for the education system as a system (Torrance, 1995, p. 45). The possible research questions will increasingly be concerned with latter impact, even though there remain traditional questions, e.g., regarding the extent to which assessments measure what they purport to measure.

Measurement-driven instruction and assessment-led curriculum development always run the risk of getting into a vicious circle. You immediately need a higher level control to check that the assessments in use have beneficial effects rather than function as cues for superficial learning and instruction. This is where the social constructivist view of education in society comes in. In the same way that the criterion of viability is applied to the identification of "good" mathematics, it can also be applied to identify "good" assessment methods. Methods that are just fads will automatically die out – society will only accept what proves reliable and helps the young ones learn more "good" mathematics.

Of course there is the question of the time scale. Do we have the time to try out a lot of new ideas which we are not sure of? Who are the people to judge what is worth trying in the way of assessment methods? Isn't it safer to be cautious and let somebody else make the mistakes, and then, on the basis of their experiences, pick up what proves useful?

The responsibility for researchers in mathematics education includes taking these kinds of ungratifying initiatives at the curricular level. Being professional does not mean being automatically right, but it includes a certain kind of leadership that should not be surrendered to others, who are, on the whole, less well equipped to judge what is good. In the same way, in their own classrooms, teachers are the professionals, selected by society, who should judge what is good for individual students.

The psychological issues that are related to the process through which society, or part of society, adopts or rejects new methods of assessment have barely been touched on yet. What are the conditions for an innovation to get a foothold? How long must the trial period be, before you can safely say that it will have a "lasting" effect? Questions like these belong to social psychology – a branch of psychology that has had few connections with mathematics education.

In my own country, Finland, grades in different subjects are given on a scale that has been nationwide and unchanged for generations. It is an integral numerical scale with 10 as the highest mark and 4 as failing. A given grade reports, in one number, all the achievements of the student, e.g., in mathematics. It is clear that such a tradition is a very strong obstacle for the introduction of modes of assessment that would require a more detailed (e.g., verbal) report of the outcome of the assessment.

## Dynamic assessment.

The issue of static versus dynamic assessment is also related to the function of assessment within the educational process. But whereas measurement-driven instruction or assessment-led curriculum development challenge the traditional relationship between assessment and the outcomes of education, dynamic assessment refers to the assessment of individuals' responsiveness to teaching (Bazzini, 1993, p. 100). These two aspects may be connected, if we identify responsiveness to teaching as an important indicator of a person's mathematical performance.

In fact, such an identification is not too far-fetched. Mathematics in present-day society is much less individual work and much more group work than it used to be, and in their future occupations many students will have to rely on their co-workers to provide information which is needed for the solution of problems. Even in lower age mathematical classrooms, mathematical talent is often identified by how quickly you catch on to a line of reasoning.

Normally, however, dynamic assessment is associated with remedial instruction in mathematics. One main feature is a systematic attempt to actively change various components of tasks and approaches to teaching in order to find conditions that are most effective for each child (Bazzini, 1993, p. 100). The measurement and remeasurement of existing capacities is abandoned in favour of first inducing and then assessing modified performance right in the test situation itself. In such assessment of modifiability one attacks the cognitive functions found to be directly responsible for the usually demonstrated deficiencies (Feuerstein, Rand & Hoffman, 1979, p. 89).

The procedure derives from a particular interpretation of Vygotsky's zone of proximal development (ZPD). One tries to assess the students' current state in relation to the zone available for, e.g., acquiring a concept. One then assesses the "modifiability" by giving a certain amount of help, being careful not to give more help than is needed (Newman, Griffin & Cole, 1989, p. 79). This gives measures of both the actual state and the "readiness to learn", which may be interpreted separately from each other.

Newman, Griffin and Cole call this "assessment *by* teaching", with the goals of teaching subordinated to the goals of assessment. They also advocate, under certain conditions, reversing the priorities utilizing "assessment *while* teaching", in which case the amount of support initially given exceeds the minimal amount needed, and decrements in the amount of support indicate an increase in the students' independent ability to carry out a task (Newman, Griffin & Cole, 1989, p. 81). The choice between the approaches depends on the purpose of the assessment, the number of students one is working with at the same time, and the degree to which the subject matter (the object of assessment) decomposes itself into a neat sequence with a preferred route to mastery.

The dynamic varieties of assessment constitute an interesting field for new innovations in assessment. While they in no way lessen the need for "static" forms of assessment, they seem to offer ways of increasing the amount of information one can get about the mental processes leading to mathematical knowledge. In an age of technical advances, computerized dynamic assessment is an obvious subfield. However, even though computers are convenient for dynamic assessment – being a mixture of assessment *and* teaching, this is an area where strong arguments may be raised that *flexibility* is needed in response to very great individual differences. In such case, teacher expertise is indispensable.

Whether teachers accept dynamic assessment as an important part of mathematics education and whether society sees any need for reports on students' readiness to learn mathematics are, of course, unanswered questions which again relate to the field of social psychology applied to mathematics education.

## Validation of assessment procedures

A researcher looking for arguments why certain assessment methods are "good", while others are not, will certainly formulate some of his or her reasoning in terms of validity. Other criteria, such as objectivity, reliability and cost, have of course to be added, but from the psychological point of view validity is a most interesting criterion.

For the moment, I will pass by some of the more troublesome validity aspects of assessment in mathematics, important though they are. It is obvious that the complex nature of mathematical knowledge makes it difficult to use tests that differ from each other with respect to their assessment modes as validation for each other. Gender issues in assessment is a good example. It was shown by Forbes (1995), in a comparison between internal assessments and a national examination in New Zeeland, that girls tend to perform relatively better on internal assessments, whereas boys do better on the written examination (with a time limit). If one uses the written examination to moderate the scores on the internal assessments, a systematic error is made which introduces a gender bias.

More to the point of the discussion, as theories of learning mathematics develop and undergo refinement, assessment methods that are based on new theoretical constructs are necessary in order to achieve correspondence between theory and practice. This type of construct validity of assessment actually can be achieved at three different levels. Stating what might be appropriate to do is one thing, knowing what to do is a different thing and actually "doing it" is yet a third one (Philippou & Christou, 1996, pp. 136-137).

In a specific study, where they were concerned with categorising students' responses as showing proceptual understanding or not, Hunter & Monaghan (1996, p. 103-104) met with considerable difficulties – the labile nature of conceptions of the students, the contextual dependence, and cases where the student's poor technical ability masked the assessment of conceptual understanding.

In another study, Morgan (1966) pointed at the theoretical contradiction, from a constructivist perspective, in assuming that scrutiny of texts produced by students in the classroom context will provide valid evidence of their mathematical thinking and attainment. It implies that meaning resides within the text, available to an ideal reader, rather than being constructed by different readers with their respective resources.

It appears that many theories of mathematics learning are too refined or philosophically complex to have indisputable practical implementations in the form of criterion-referenced assessment. A solution might be to use the kind of construct-referenced assessment (Wiliam, 1994, p. 59) that one also needs when the performance is not reducible or not definable. The common feature is that one does not expect immediate correspondence between theory and practice. Rather, the process of validation is placed in the social domain. The practical procedure is, e.g., to have a group of assessors come to an agreement, in an iterative way, whether a performance corresponds to theory (or whether a student has a specific kind of conceptual understanding). Experiences show that assessor training is important, but the process usually leads to good agreement even if the shared theoretical constructs of the assessors are unclear. It should be noted that this is a perfect example of social construction of (assessor) knowledge – it also points at the possibility of improving theory in light of the outcome of the social process.

It is interesting to note that the concept of validity has been undergoing an evolution that reflects or parallels the changing view of the function of assessment. Whereas formerly validity referred to the traditional purposes of assessment – gathering of information or making decisions, it is now not uncommon to talk about *consequential validity*, which refers to intentional shaping of social reality. A method of assessment is valid in this sense, e.g., if it influences the way teachers and students think or act in an educational process, in accordance with expectations.

It is difficult to discuss consequential validity without bringing in ethics or at least reflecting over the justification of intentional shaping of social reality. The responsibility tied to assessment is comparable to the responsibilities connected with teaching or making curricular decisions – in fact, consciousness of the consequences of assessment definitely makes assessment an integral part of the educational process. Accepting that you cannot isolate assessment from the rest of the process, one would expect decision makers with the power to prescribe nation-wide assessment schemes to think twice and base their initiatives on research that includes studies of potential multi-level effects of the proposed assessment schemes.

A particular example is to be found in the presently common strivings for "authentic assessment". There are good arguments why assessment schemes should be developed in the direction of more authentic assessment in mathematics, not in the least with respect to furthering the attitudes towards mathematics. But unless other conditions support such a change (which may be a substantial one), it may have other consequences that are negative. In Finland, students at the lower secondary school level have only three 45-minute lessons of mathematics per week. Teachers are already hard pressed for time. They do not see how to fit in time-consuming new methods of assessment, let alone preparing their students for such assessments. Under these circumstances the change easily becomes a superficial one, with mediocre teaching of "authentic mathematics" that will not fill other criteria of validity. Add to that some desillusioned teachers.

This is, obviously, not a critique of teachers, but rather a reminder that a great variety of facts have to be taken into account when assessment schemes are introduced with the intention to cause a change. We all know a lot about human reactions. Why are they not more often treated as an important variable in psychological research in mathematics education?

## Teacher concepts of assessment

The social constructivist perspective on education puts a strong emphasis on the role of the teacher in the process by which the students construct their knowledge, not only as a facilitator of learning, but as a person with a social responsibility (Björkqvist, 1995). The teacher is the representative selected by society to keep the culture of mathematics available to the students. Considering the changing nature of collective knowledge and the conditions of society, this is not a definite, but a provisional position of authority – a matter of confidence in a professional. The picture includes the expectation that the teacher is capable of placing the conceptual world of a student in a developmental perspective better than anybody else.

The student tests his or her conceptions primarily on the teacher, and gradually, with the support of the teacher, begins to practice self-assessment. The teacher analyses the conceptions and attempts to predict the effects when they are confronted with other ways of thinking that may predominate in society. Part of the teacher role is promoting the development of metacognition, making the student aware of his or her own thinking and that it may differ from that of others.

The teacher also is the one to make decisions in cases where different criteria for the viability of knowledge are to be balanced against each other, e.g., individual creativity versus precision in conventional terminology.

It is expected that the teacher has visions of the social development of mathematics and that those visions are actually put into use in the planning of educational activities and in the assessment of the outcomes of them.

It is remarkable that the social constructivist perspective, while firmly based on the view that each individual constructs his or her own knowledge, should imply such an important role for the teacher in that process.

Therefore, it is of utmost importance that as much as possible is known about the teachers' own views about their part in the process, whether they see themselves in a way that is congruent or inconsistent with the ideal picture. And if there are inconsistencies, of what kind are they?

Some interesting research was conducted by the late Stieg Mellin-Olsen with a focus on the way Norwegian teachers viewed the structure of school mathematics and the progress of their students in mathematics (Mellin-Olsen, 1990). He found that they frequently referred to mathematics using the metaphor of a journey along a track, sometimes with greater speed and sometimes more slowly. The track seemed often to be formed by the tasks in a mathematics textbook – in fact, this was so common that the discourse of school mathematics in many cases was a "task discourse". The associated view of school mathematics is quite far from Pimm's (1996, p. 37) description of it as a complex interlocking system of propositional knowledge, concepts, problems, techniques, methods, working practices, communities, traditions and beliefs. There is no reason to expect that the situation is unique to Norway.

One should then not be surprised if teachers hold widely different views of the functions of assessment in mathematics. The conceptual framework in this respect has been studied by, e.g., Rico et al. (1995), who attempted to create a system of categories which shape mathematics teachers' ideas and concepts about assessment. Such a system is important because it can provide a foundation for comparisons of studies on teachers' views of assessment in mathematics. To monitor changes in those views, the theoretical basis for the categories would have to have a strong component of social psychology.

Knowledge why there exist discrepancies between society's goals for mathematics education and the implemented modes of assessment is important as such, but if one wants to impose change towards greater congruence, one needs a theoretical foundation that goes beyond traditional mathematics education and includes mechanisms of influence on the practice of teachers in an explicit way.

There are already indications that participation in projects that develop new methods of assessment in some cases gives lasting effects – also with respect to the teachers' appreciation of the theory behind the methods. Yackel, Cobb & Wood (1992, p. 77) remark that the most striking thing that they found in their informal interviews of participating teachers was "the radical change in the very basis on which they assess individual children's learning".

However, the situation is not by far the same with the views and actions of teachers in non-privileged positions. The available mechanisms of influence are less well known and probably quite different. Teachers, like researchers, come with their own background marked by their mathematical experiences and the way they feel society treats them. In-service training, e.g., has to acknowledge this fact and should ideally be individualised.

To see things in new ways, one needs events of "coming-to-see" (Mason, 1994). Very often this involves the removal of an *obstacle* in the form of a habit of mind. Morgan (1994) gives a relevant example of such an obstacle that relates to assessment in mathematics.

> One aspect that is completely missing from the interviews is any perception of algebraic symbolisation as a tool for creating further mathematics or for solving problems. For teachers as well as for the students, writing an algebraic formula is the solution.
>
> [Morgan, 1994, p. 301]

Similarly, to be able to act in new ways, one needs to eliminate obstacles in the form of social habits, or get help with the removal of them. The easy way out is to introduce new routines, e.g., in the form of standardised versions of new modes of assessment. Morgan (p. 302) criticizes this as a distortion of the original intentions, with which it is easy to agree. There may, however, be cases where access to standardised versions is the key to new habits that need not be mechanical applications of those standardised versions – especially if the principal obstacle for the teacher is lack of time, while the mental readiness for change is there.

Research on the implementation of new modes of assessment does not have to be framed in terms of obstacles. Another possibility is using the notion of *sociomathematical norms*, introduced by Yackel and Cobb (1995) as a conceptual framework for talking about, describing, and analysing the mathematical aspects of teachers' and students' activities in the classroom. They include normative understandings of what counts as a different solution, a sophisticated solution, an efficient solution, an explanation, or a justification.

> Sociomathematical norms are intrinsic aspects of the classroom's mathematical microculture. ... [T]hese norms are not predetermined criteria introduced into the classroom from the outside. Instead, they are continually regenerated and modified by the students and the teacher through their ongoing interactions.
>
> [Yackel & Cobb, 1995, p. 270]

In the case of assessment in mathematics, sociomathematical norms exist and evolve not only at the classroom level, but also at a level where teachers interact with each other, with political decision makers, and the rest of society.

This is where understandings are shaped regarding what counts as a qualitatively different assessment method, a valid assessment method, an efficient assessment method, etc.

Making the sociomathematical norms explicit serves the purpose of expediting this process of negotiation, and it also helps the teachers find their professional authority.

## Conclusion

In my presentation of some psychological issues that relate to assessment in mathematics education, I have spent rather little time on the relationship between attitudes to mathematics and performance in mathematics. This area, however, has been the subject of much PME research, including analyses of the psychological mechanisms, in both directions, that may serve to explain observed correlations (e.g., Minato & Kamada, 1992).

I have felt it more appropriate to focus on social issues that refer to the changing views of school mathematics and the poor alignment between the goals and the teaching methods on one hand, and the purposes and modes of assessment on the other. I have attempted to show that, from several different perspectives, one comes to the conclusion that the framework for research could use more influence from social psychology.

This does not mean less emphasis on mathematics. In fact, it remains likely that the arguments in the social negotiation of the purposes and modes of assessment in mathematics will continue to be mathematical – but the visions of what is viable school mathematics will change.

## References

Balacheff, N. (1990). Future Perspectives for Research in the Psychology of Mathematics Education. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition*. Cambridge: Cambridge University Press.

Bazzini, L. (1993). The Teaching/Learning Process and Assessment Practice: Two Intertwined Sides of Mathematics Education. In M. Niss (Ed.), *Cases of Assessment in Mathematics Education: An ICMI Study*. Dordrecht: Kluwer Academic Publishers.

Bjorkqvist, O. (1993). Social konstruktivism som grund för matematikundervisning. [Social Constructivism As a Foundation For the Teaching of Mathematics]. *Nordisk matematikkdidaktikk 1*(1): 8-17.

Bjorkqvist, O. (1995). Assessment in Mathematics from a Social Constructivist Viewpoint. In: E. Pehkonen (Ed.), *NORMA 94 Conference. Proceedings of the Nordic Conference on Mathematics Teaching in Lahti 1994*. Helsinki: Department of Teacher Education, University of Helsinki. Research Report 141.

Clarke, D. J. & Sullivan, P. A. (1992). Responses to Open-Ended Tasks in Mathematics: Characteristics and Implications. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth PME Conference* (Vol. I, pp. 137-144), Durham, NH, USA.

Ernest, P. (1991). *The Philosophy of Mathematics Education.* Basingstoke: The Falmer Press.

Feuerstein, R., Rand, Y. & Hoffman, M. B. (1979). *The Dynamic Assessment of Retarded Performers.* Baltimore: University Park Press.

Forbes, S. D. (1995). Are Assessment Procedures in Mathematics Gender-Biased? In B. Grevholm & G. Hanna (Eds.), *Gender and Mathematics Education, an ICMI Study.* Lund: Lund University Press.

Hunter, M. & Monaghan, J. (1996). Some Issues in Assessing Proceptual Understanding. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 97-104), Valencia, Spain.

Jaworski, B. (1994). The Social Construction of Classroom Knowledge. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 73-80), Lisbon, Portugal.

Krainer, K. (1992). Powerful Tasks: Constructive Handling of a Didactical Dilemma. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth PME Conference* (Vol. II, pp. 25-39), Durham, NH, USA.

Mason, J. (1994). Researching From the Inside in Mathematics Education – Locating an I-You Relationship. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education* (Vol. I, pp. 176-194), Lisbon, Portugal.

Mellin-Olsen, S. (1990). Oppgavediskursen [The Task Discourse]. In G. Nissen & J. Bjørneboe (Eds.), *Matematikundervisning og Demokrati. Rapport fra en konference afholdt 14.-16. Juni 1990 i Gilleleje.* Roskilde, Denmark: IMFUFA, Roskilde Universitetscenter.

Minato, S. & Kamada, T. (1992). Results of Researches on Causal Predominance Between Achievement and Attitude in Junior High School Mathematics of Japan. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth PME Conference* (Vol. II, pp. 112-119), Durham, NH, USA.

Morgan, C. (1994). Teachers Assessing Investigational Mathematics: The Role of "Algebra". In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 295-302), Lisbon, Portugal.

Morgan, C. (1996). Language and Assessment Issues in Mathematics Education. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 19-26), Valencia, Spain.

Newman, D., Griffin, P. & Cole, M. (1989). *The Construction Zone. Working for Cognitive Change in School.* Cambridge: Cambridge University Press.

Niss, M. (1993). Assessment in Mathematics Education and Its Effects: An Introduction. In M. Niss (Ed.), *Investigations into Assessment in Mathematics Education. An ICMI Study.* Dordrecht: Kluwer Academic Publishers.

Philippou, G. & Christou, C. (1996). Teacher's Conceptions About Mathematical Assessment. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 131-138), Valencia, Spain.

Pimm, D. (1996). Modern Times: The Symbolic Surfaces of Language, Mathematics and Art. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 35-50), Valencia, Spain.

Rico, L., Fernández, F., Gil, F., Castro, E., Castro, E., del Olmo, A., Moreno, F. & Segovia, I. (1995). Teacher's Conceptual Framework on Mathematical Assessment. In D. Carraher & L. Meira (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 130-137), Recife, Brazil.

Schliemann, A. D. (1995). Some Concerns About Bringing Everyday Mathematics to Mathematics Education. In D. Carraher & L. Meira (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 45-60), Recife, Brazil.

Silver, E. (1993). On Mathematical Problem Posing. In I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin (Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 66-85), Tsukuba, Japan.

Torrance, H. (1995). Teacher Involvement in New Approaches to Assessment. In H. Torrance (Ed.), *Evaluating Authentic Assessment.* Buckingham: Open University Press.

Webb, N. L. (1993). Assessment for the Mathematics Classroom. In N.L. Webb & A. F. Coxford (Eds.), *Assessment in the Mathematics Classroom. 1993 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics.

Wiliam, D. (1994). Assessing Authentic Tasks: Alternatives to Mark-Schemes. *Nordic Studies in Mathematics Education 2*(1), 48-68.

Yackel, E., Cobb, P. & Wood, T. (1992). Instructional Development and Assessment from a Socioconstructivist Perspective. In G. Leder (Ed.), *Assessment and Learning of Mathematics.* Hawthorn, Australia: The Australian Council for Educational Research.

Yackel, E. & Cobb, P. (1995). Classroom Sociomathematical Norms and Intellectual Autonomy. In D. Carraher & L. Meira (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 264-271), Recife, Brazil.

# NEUROMAGNETIC APPROACH IN COGNITIVE NEUROSCIENCE

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*To increase our understanding of the functional organization of the human brain, the sequence of cortical events related to sensory analysis of repeated stimuli or to a performance of specific tasks needs to be characterized and quantified with a good temporal and spatial resolution. Magnetoencephalography provides a totally noninvasive method to study functions of the living human brain. With whole-scalp neuromagnetometers one can monitor the activation of several cortical areas simultaneously and accurately in time and space.*

## Introduction

The challenge of cognitive neuroscience is to describe the relationship between the brain and the mind, *i.e.*, to reveal how structural neural elements are driven into the physiological activity that results in perception and cognition. Although animal studies are useful for the understanding of brain anatomy and physiology, many fundamental questions about human information processing are still unknown. The information flow in the brain during the performance of complex mental tasks cannot easily be described by anatomical or physiological studies of animals. The neural correlates of higher mental functions, such as language or mathematical reasoning, need to be sought directly in awake humans.

The last few decades have witnessed an explosion of knowledge concerning how the human brain works under normal and pathological conditions. One of the major reasons for this has been the development of novel brain imaging techniques that permit one to study noninvasively the neural processes that were previously accessible only in experimental animals and brain-injured patients. Brain imaging techniques can be divided into two broad categories: structural and functional. The two major structural techniques are computerized tomography (CT) and magnetic resonance imaging (MRI) which both provide accurate information about the anatomical structures of the human brain on a millimeter scale. Functional imaging techniques allow us to observe the human brain in action and thus to identify the brain regions that are involved when the subject receives specific sensory stimuli or performs a given task. Positron emission tomography (PET) and functional magnetic resonance imaging (fMRI) can trace activity changes in the human brain at millimeter precision but the temporal resolution of these techniques is limited to tens of milliseconds or even to few seconds. Thus the sequence of activation of distinct brain regions involved in the sensory analysis of various stimuli or in the performance of a given tasks cannot easily be followed with PET or fMRI recordings. The advantage of magnetoencephalography (MEG) and electroencephalography (EEG) over PET and fMRI lies in their excellent temporal resolution: both have a temporal resolution of a millisecond scale. Since magnetic signals can be picked up outside the head undisturbed by tissue inhomogeneities and MEG typically sees only a part of the whole cortex activity, the interpretation of the measured MEG signal patterns is more straightforward than that of the corresponding EEG distributions.

The whole-scalp 122-channel magnetometer (Neuromag-122$^{TM}$), developed at the Helsinki University of Technology by Neuromag Ltd. provides us with a powerful tool to study the neural basis of human cognitive functions, since activation of several cortical areas can be simultaneously detected with good temporal and spatial resolution. With the neuromagnetic method one can study human cortical reactivity by collecting evoked responses triggered by various sensory stimuli and by recording spontaneous activity during different tasks.

## Detection of neuromagnetic signals

Electric currents flowing in the brain generate weak magnetic signals, typically 100–1000 fT (fT = femtoTesla = $10^{-15}$ T), which is only one billionth of the Earth's static magnetic field. Due to the small size of cerebral magnetic signals, measurements are made in a magnetically shielded room. Figure 1 illustrates recordings with the 122-channel magnetometer located in the magnetically shielded room at the Helsinki University of Technology. Signals are first picked up by superconducting flux transformers and then detected by SQUIDs (Superconducting QUantum Interference Devices) that are extremely sensitive to small changes in the magnetic fields (see Fig. 1).

This Neuromag-122™ instrument houses 122 planar first-order gradiometers, located at 61 measurement sites (Ahonen et al. 1993). Each sensor unit contains a pair of gradiometers that measure the two orthogonal tangential derivatives ($\partial B_r/\partial x$ and $\partial B_r/\partial y$) of the magnetic field component ($B_r$) normal to the sensor surface at the sensor location. Such sensors pick up the largest signal just above a local source, where the field gradient has its maximum. The two independent derivatives measured at each site also give an estimate of the direction of the local source current (for a review, see Hämäläinen et al. 1993).

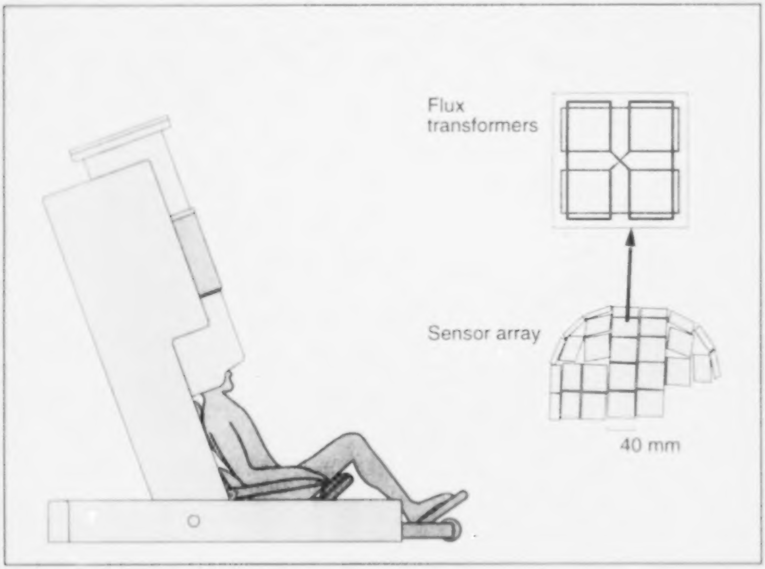

**FIG. 1 Left:** Schematic illustration of the MEG installation at the Helsinki University of Technology. Measurements are carried out in the magnetically shielded room. The concave, helmet-shaped bottom of the helium dewar, with the flux-transformer coils near its tip, is brought as close as possible to the subject's head. The instrument employs 122 planar first-order gradiometers, located at 61 measurement sites. **Right:** At each of the 61 recording locations of the sensor array (insert at lower right), two orthogonal derivatives of the magnetic field component normal to the sensor surface are measured simultaneously. The planar flux transformers of the device have a figure-of-eight configuration (insert at upper right).

## Neural origin and interpretation of magnetic signals

Information transfer within and between neurons results in small electric currents which are associated with electric potentials and magnetic fields. Signals are mediated by action potentials, which propagate along axons and trigger the release of neurotransmitters in chemical synapses. Release of transmitters opens selective ion channels in the membrane of the postsynaptic neurone, and thereby generates postsynaptic potentials (PSPs), either excitatory (EPSPs) or inhibitory (IPSPs), which cause a current flow along the interior and exterior of the cell.

The cerebral magnetic fields reflect mainly postsynaptic currents in cortical neurons, and the direction of the external magnetic flux is determined by the direction of the net intracellular current flow according to the right-hand rule (Hari 1990, 1993; Hämäläinen et al. 1993). When the postsynaptic currents in thousands of nearby pyramidal cells within a few square centimeters of the cortex act in shynchrony, the current is strong enough to be detected outside the scull.

A crude estimate for the dominant source area of an evoked response can be obtained directly from the distribution of the MEG signals measured with planar gradiometers: a local signal maximum typically suggests cortical current flow beneath that sensor. More sophisticated methods are needed to obtain spatially and temporally accurate information about the distribution of the underlying current sources. However, since the inverse problem, *i.e.*, the deduction of the underlying electric brain activity from the extracranial magnetic field, has no unique solution, characterization of the source currents requires *a priori* assumptions. The constraints of the model can be based either on the anatomy (all currents are constrained to the cortex) or the physiology (unphysiological current strenghts can be rejected) of the brain (Hari 1996).

The simplest source model is a current dipole. The equivalent current dipole (ECD) best explaining the measured field pattern is found by a least-squares search using the measured signals at a selected time instant. This results in a dipole representing the 3-dimensional location, orientation, and strength of current flow in the activated cortical site. However, after a single stimulus, several brain areas may be active simultaneously and therefore multidipole models are often needed to interpret complex field patterns. More precise information about the source locations with respect to anatomical structures can be found by coupling MEG data with MRI data.

The spherical volume conductor model of the head, with the parameters fitted according to the individual brain anatomy, works well with most applications but a more realistic head model is necessary to better describe those parts of the head, *e.g.*, bottom of the brain and the temporofrontal scull base, that deviate from the spherical symmetry. In a sphere, currents that have a component tangential to the surface of the sphere produce an external magnetic field whereas radial currents, *i.e.*, currents perpendicular to the sphere surface, do not (for a review, see Hämäläinen et al. 1993). Because the sensitivity of MEG decreases rapidly as the source depth increases, MEG is most suitable for studies of human cortical functions. MEG signals are mainly produced by currents flowing in the fissural cortex, where the apical dendrites of pyramidal cells, oriented along the cortical columns, are tangential to the skull.

## Evoked fields

With evoked response recordings it is possible to study the activation of distinct cortical areas by various sensory stimuli. Stimulus repetition and signal averaging are usually needed to make the signals discriminable from the spontaneous activity of the brain, and from the instrumental and environmental noise. The whole-scalp MEG recordings have revealed several source areas that would have been difficult to detect with smaller sensor arrays.

### *Auditory responses and sensory memory*

The source locations of consecutive deflections of the auditory evoked magnetic field vary slightly, implying that the center of gravity of the activation changes as a function of time between different cytoarchitectonic areas of the auditory cortex. Sensory memory in its simple form can be studied by determining recovery cycles of different MEG responses. The most prominent signal, the N100m peaks at about 100 ms after the tone onset or a change within a tone. The N100m amplitude increases with the interstimulus interval (ISI) and reaches a plateau at ISIs of about 8 s (Hari et al.1982). The ISI dependence of N100m amplitude is similar in both hemispheres, suggesting that tones leave traces of similar duration into auditory cortices of both hemispheres (Mäkelä et al. 1993).

Several MEG studies have documented the sensitivity of the auditory cortex to various changes in the sensory input (for reviews, see Hari 1990; Näätänen et al. 1994). One way to study these processes is to present the stimuli in an oddball manner: a monotonous sequence of "standard" tones is interrupted from time to time by a physically "deviant" tone. The right auditory cortex, and some other brain areas, seem to react more strongly than the corresponding areas in the left hemisphere to various changes in the sound environment (Leyänen et al. 1996; Raij et al. 1997).

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Figure 2 shows auditory evoked magnetic signals to left- and right-ear deviants that differ from the standards in duration. In addition to the N100m deflection, responses to deviants contain clear deflections at about 200 ms in each hemisphere. The peak amplitude of this magnetic mismatch response, MMF, is larger in amplitude and more widely distributed in the right than in the left hemisphere, irrespective of the ear of stimulation.

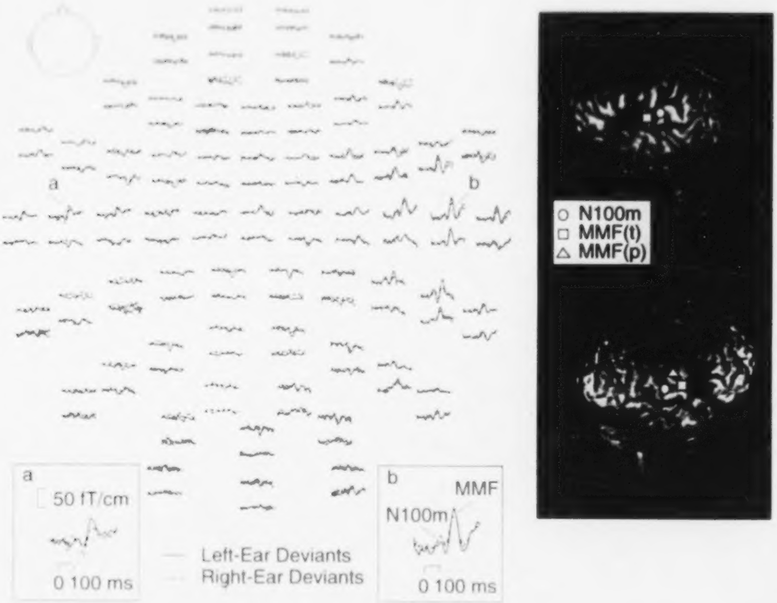

**FIG 2. Left:** A 122-channel recording of auditory evoked magnetic fields to left- and right-ear duration deviants. The helmet is viewed from above, and the upper and lower traces of each response pair show the latitudinal and longitudinal derivatives of the magnetic filed, respectively. The insert shows enlarged MMF responses recorded over the left and right temporal areas. **Right:** The sources of the N100m to standards and MMFs to duration deviants are superimposed on the same subject's MRI surface renderings viewed from the left (upper) and right (lower). The N100m and the MMF(t) sources were in the supratemporal cortex but have been projected to the brain surface for better visualization. The MMF(p) source is in the inferoparietal cortex. (Adapted from Levänen et al. 1996.)

Investigations of the signal distributions measured with the whole-scalp neuromagnetometer have revealed that infrequent deviants evoke bilateral MMFs with sources in the supratemporal auditory cortex slightly anterior to the N100m source to the standards (Levänen et al. 1996). The source locations of these supratemporal MMFs vary in a feature-specific manner, suggesting that the standards leave distributed neuronal representations in the human auditory cortex (Levänen et al. 1993, 1996).

In addition, various deviants activate the same inferoparietal area in the right hemisphere (see Fig. 2), irrespective of the stimulated ear (Levänen et al. 1996). This right-hemisphere component, activated at about 250 ms after the stimulus onset, probably reflects more global auditory change detection and is more closely related to the orienting reflex and attention than the feature-specific supratemporal components. These results suggest parallel processing of auditory stimulus features and stronger involvement of the right than the left hemisphere in memory-based processing of simple tones.

A random omission of some sounds of a monotonous sequence is a dramatic change of stimulation. When omissions are attended by the subject, they produce strong responses which peak 145–195 ms after the predicted onset of the sound and are seen over both hemispheres (Raij et al. 1997). A major part of omission signals appears to be generated in the supratemporal auditory cortex, implying that the sensory specific projection cortex keeps a track of complex changes in the environment. The omissions also activate the superior temporal sulcus and the frontal lobe, again more prominently in the right than in the left hemisphere (Raij et al. 1997).

### *Somatosensory responses*

Electric stimulation of the median or ulnar nerve evokes a synchronous action potential burst which reaches the contralateral primary somatosensory cortex, SI, within 20 ms. The sites of earliest SI activity are somatotopically organized and the course of the central sulcus can be easily determined by stimulation of various body regions. Whole-scalp neuromagnetic recordings after electric stimulation of upper limb nerves at long ISIs imply activation of several other source areas in addition to SI. Source modelling has revealed that activation of the contralateral SI cortex is followed by bilateral activation of SII cortices and by contralateral activation of the posterior parietal cortex (Forss et al. 1994). Recently, Forss et al. (1997) observed additional signals in the mesial cortex, when the subject attended to randomly presented, infrequent ulnar and median nerve stimuli.

Differentiation of signals generated in these distinct brain areas allows monitoring of activity changes in association with different tasks and sensory inputs. The specific part of SI cortex, assumed to be the generation site of the somatosensory 30-ms response, is considered to receive strictly contralateral afferentation. However, the 30-ms response to median nerve stimuli is suppressed in the contralateral SI but enhanced in the ipsilateral S1 when tactile stimulation is simultaneously delivered to one hand only (Schnitzler et al. 1995), suggesting that tactile input from one hand has access to the SI cortices of both hemispheres, probably through callosal connections.

## Spontaneous cortical rhythms

Spontaneous cortical rhythms most probably reflect a complex interplay in circuits that exist between different parts of the cortex and between the cortex and subcortical brain areas, but the functional significance of these rhythms has remained largely unknown. Since spontaneous rhythms do not repeate themselves from time to time, its is important to be able to record them simultaneously over various distinct brain areas. Recordings with the whole-scalp neuromagnetometer have shown that several regions of the healthy human cortex have their own intrinsic rhythms, typically between 8–40 Hz in frequency, and with modality- and frequency-specific reactivity (for a review, see Hari and Salmelin 1997).

### *Parieto-occipital rhythm*

Sources of the posterior 10 Hz alpha rhythm concentrate predominantly in the parieto-occipital reagion with less activity in the occipital cortex. The level of the alpha activity decreases during visual stimulation. However, the suppression occurring during passive viewing of pictures is considerably intensified when the subject is asked to name the pictures, either silently or aloud (Salmelin et al. 1994). Salenius et al. (1995) showed that changes in the alpha activity produced by visual imagery are very similar to those elicited by actual visual stimuli, suggesting that the visual areas close to the parieto-occipital sulcus are involved in the generation of mental images.

### *Somatomotor rhythms*

Spontaneous MEG signals over the somatomotor regions have dominant frequencies around 10 and 20 Hz. Both of these frequency components, originating close to the primary somatomotor hand area, react with a rebound to the offset of a brisk movement but the rebound is slightly faster and clearly stronger for the 20-Hz than for the 10-Hz signals (Salmelin and Hari 1994). Since the sources for the 20-Hz component are anterior to those for the 10-Hz component, the 20-Hz component has been assumed to receive contribution from the precentral motor cortex. The strong rebound of the 20-Hz activity occurring after electric stimulation of the median nerve at the wrist is abolished by simultaneous exploration of objects with the fingers of the stimulated hand. Interestingly, the rebound is also strongly suppressed when the subject imagines exploratory finger movements (Schnitzler et al. 1995), suggesting involvement of the primary motor cortex in motor imagery.

## Concluding remarks

Whole-head magnetoencephalographic recordings during various simple stimulations and tasks have considerably added to our basic understanding of the functional organization of the human cerebral cortex. Now it is possible to progress towards the use of stimuli and tasks that resemble more closely real-life situations. Perhaps, in the future, it will also be possible to describe how the brain enables the mind.

## References

Ahonen A, Hämäläinen M, Kajola M, Knuutila J, Laine P, Lounasmaa OV, Parkkonen LP & Tesche C (1993) 122-channel SQUID instrument for investigating the magnetic signals from the human brain. *Physica Scripta* T49: 198-205.

Forss N, Hari R, Salmelin R, Ahonen A, Hämäläinen M, Kajola M, Knuutila J & Simola J (1994) A novel source area in the human parietal cortex activated by median nerve stimulation. *Experimental Brain Research* 99: 309–315.

Forss N, Merlet I, Vanni S, Hämäläinen M, Mauguiere F & Hari R (1997) Activation of human mesial cortex during somatosensory attention task. *Brain Research* . In press.

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Hämäläinen M, Hari R, Ilmoniemi R, Knuutila J & Lounasmaa OV (1993) Magnetoencephalography - theory, instrumentation, and applications to noninvasive studies of the working human brain. *Reviews of the Modern Physics* 65: 413-497.

Hari R (1990) The neuromagnetic method in the study of the human auditory cortex. In *Auditory Evoked Magnetic Fields and Potentials. Advances in Audiology* (F Grandori, M Hoke & GL Romani, Eds) Basel: Karger, pp 222-282.

Hari R (1993) Magnetoencephalography as a tool of clinical neurophysiology. In *Electroencephalography: Basic Principles, Clinical Applications, and Related Fields* (E Niedermeyer & F Lopes da Silva, Eds) Baltimore: Williams & Wilkins, pp 1035-1061.

Hari R (1996) MEG in the study of human cortical functions. In *Visualization of Information Processing in the Human Brain: Recent Advances in MEG and Functional MRI* (I Hashimoto, YC Okada & S. Ogawa, Eds) Elsevier Science B.V. EEG Supplement 47, pp 47–54.

Hari R & Salmelin R (1997) Human cortical oscillations: a neuromagnetic view through the skull. *Trends in Neuroscience* 20: 44–49.

Hari R, Kaila K, Katila T, Tuomisto T & Varpula T (1982) Interstimulus-interval dependence of the auditory vertex response and its magnetic counterpart: Implications for their neural generation. *Electroencephalography and clinical Neurophysiology* 54: 561-569.

Levänen S, Hari R, McEvoy L & Sams M (1993) Responses of the human auditory cortex to changes in one *versus* two stimulus features. *Experimental Brain Research* 97: 177-183.

Levänen S, Ahonen A, Hari R, McEvoy L & Sams M (1996) Deviant auditory stimuli activate human left and right auditory cortex differently. *Cerebral Cortex* 6: 288-296.

Mäkelä JP, Ahonen A, Hämäläinen M, Hari R, Ilmoniemi R, Kajola M, Knuutila J, Lounasmaa OV, McEvoy L, Salmelin R, Salonen O, Sams M, Simola J, Tesche C & Vasama J-P (1993) Functional differences between auditory cortices of the two hemispheres revealed by whole-head neuromagnetic recordings. *Human Brain Mapping* 1: 48-56.

Naatanen R, Ilmoniemi RJ & Alho K (1994) Magnetoencephalography in studies of human cognitive brain function. *Trends in Neuroscience* 17: 389–395.

Raij T, McEvoy L, Makela JP & Hari R (1997) Human auditory cortex is activated by omissions of auditory stimuli. *Brain Research* 745: 134–143.

Salenius S, Kajola M, Thompson WL, Kosslyn S & Hari Riitta (1995) Reactivity of magnetic parieto-occipital alpha rhythm during visual imagery. *Electroencephalography and clinical Neurophysiology* 95: 453–462.

Salmelin R & Hari R (1994) Characterization of spontaneous MEG rhythms in healthy adults. *Electroencephalography and clinical Neurophysiology* 91: 273–248.

Salmelin R, Hari R, Lounasmaa OV & Sams M (1994) Dynamics of brain activation during picture naming. *Nature* 368: 463–465.

Schnitzler A, Salenius S, Salmelin R, Jousmaki V & Hari R (1995) Involvement of primary somatomotor cortex in motor imagery: a neuromagnetic study. In *Society for Neuroscience, Volume of Abstracts*, Part 1: 518.

Schnitzler A, Salmelin R, Salenius S, Jousmaki V & Hari R (1995) Tactile information from the human hand reaches the ipsilateral primary somatosensory cortex. *Neuroscience Letters* 200: 25–27.

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# DILEMMAS IN THE PROFESSIONAL EDUCATION OF MATHEMATICS TEACHERS

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*The notion of educational change is an important concept in most mathematics education courses, but a focus on change raises many problematic issues. This paper describes our use of pedagogical dilemmas as a central element of a multimedia resource for learning about mathematics teaching. The dilemmas are used to provoke detailed lesson analysis and to perturb users' assumptions about teaching and learning.*

## Ownership of educational change

In education, change is constant. Many mathematics teacher education programs focus on the notion of individual and institutional, systemic change. It has been argued that change is desirable, that change can by effected through modelling of alternate pedagogical approaches (e.g. Bednarz, Gattuso, & Mary, 1996), that change is possible but difficult (Levenberg & Sfard, 1996), and that collaboration and teacher collegiality facilitate change (Chazam, ben Chaim, & Gormas, 1996). It is assumed that changes in practice relate to changes in beliefs (e.g. Peluso, Becker, & Pence, 1996), although at least some commentators (such as Guskey, 1986) believe that changes in practice precede changes in orientation. Corsine (1987) identified expressing emotions, experiencing communalism, inculcation of hope, cognitive factors, and feedback as factors that are commonly associated with behavioural change.

A key issue that is not often reported in studies of educational change relates to the initial impetus and motivation for change. In essence, this is about ownership of change. We generally expect that improved mathematics teaching will result from reflection on practice, but while reflection is a necessary condition for on-going development and change (Schön, 1983), it seems that it is not sufficient. We have plentiful evidence around us of intelligent reflective mathematics teachers repeating inadequate practice over and over without even contemplating the necessity for or the desirability of change. This provides a dilemma for teacher educators. While, on one hand, it is clear that reflective practice cannot be mandated, and change cannot be imposed externally without significant angst, on the other hand it appears that change is more likely to occur when there is an external stimulus. Unless personal constructs are challenged, teachers are less likely to see any need for change.

Aspects of current philosophies of mathematics education, such as the metaphor of the hermeneutic circle, the notion of historicity, and belief in the individual construction of knowledge all reinforce the need for a perturbation prior to change. Thus teacher education programs need to find ways to perturb students' existing conceptions of mathematics teaching and learning, as well as the wider contexts of schooling and society, to create a milieu in which change is a desired state. This needs to be done, however, in ways that retain students' control over the content, direction and pace of change.

Clark and Peterson (1986) posed a model of teacher thinking in which teachers' thought processes, classroom constraints and their actions interact reflexively on each other. A development of this model, applied to teacher education is:

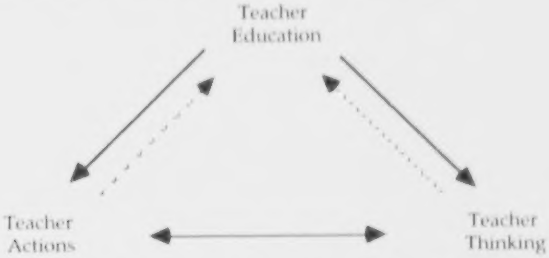

*Figure 1:* The interaction of teacher education with teacher actions and thinking

The solid lines indicate the need for challenge in the direction indicated. In other words, one of the roles of teacher education is to create an environment in which challenge of existing thinking and actions is possible. The extent to which the challenge is owned by teachers needs to be balanced against the tendency to view evidence only though limited experience in a way that reinforces current practice and so reduces the likelihood of self initiated change. The interaction between thinking and actions which is also part of the model is itself one of the goals of teacher education..

The dotted lines suggest limited effect. It is plausible that there are teacher education programs which are influenced directly by both teacher actions and teacher thinking, but even in such programs we suspect that the impact would be subtle and gradual.

This notion of challenge is elaborated further later in this paper, in a discussion about the use of technology to raise classroom dilemmas.

## Ownership of teacher education

Within a context where goals of designers of teacher education programs include changing practical constructs and actions of the participants, there are dilemmas in the choice of teaching and learning methods. Related to these are program assumptions on the meaning of professional education and the question of whether we want students to study our curriculum in their ways or to develop ways to study their own curriculum.

In conventional programs, students study lecturer-determined curricula through processes chosen by lecturers. Where there is an intention to shift from information gathering to knowledge building (c.f. Anderson & Alagumalai, 1996), there is a need for some student autonomy in the selection of learning foci, and opportunities for students to have significant control over the methods used. A suitable metaphor can be borrowed from the environmental science educators who contrast an orientation of education *about* the environment with that of education *in* the environment (Queensland Board of Teacher Registration, 1993). Mathematics teacher education is generally *about* teaching mathematics. Commonly, pre-determined curricula are established and students proceed through the various aspects of those curricula sequentially. There are undoubtedly aspects of teacher education (just as there are in the teaching of mathematics) for which this approach is desirable. Yet there are other aspects of teacher education for which education *in* the profession is more appropriate. One dilemma is to identify which aspects of professional education are better suited to the respective approaches, although it seems clear that those program elements that are related to the developing relationship between classroom practice and teacher thinking are likely to be better studied *in* context.

Education *in* the profession is unlikely to be fostered optimally through programs where the student is a passive recipient of information delivered in lectures and tutorials. Mathematics education students need to become, to some extent, autonomous operators within professional environments, taking responsibility for their own education and developing skills for life-long professional learning. This move towards autonomous learning has implications not only for the style of delivery but also for the substance and the impact of educational programs (Morgan, 1996). The orientation of students to participation is a key factor: autonomy[1], in this sense, implies less reliance on direct instruction and operative direction. It implies more student-based decision-making and control over curriculum, both in what is learned and how it is learned. Autonomy is not a function of hands off curriculum design but relates to genuine freedoms given to students (Bruce, 1995).

This movement towards autonomous learning is linked to methods of assessment. If students believe that their grades in the teacher education program are determined by their responses to a written examination, for example, it is conceivable that they will apply conventional modes, such as rote memorising of what they see as the key points, no matter how much control they have over other aspects of learning. It is possible that their learning will even be less effective than in conventional programs since there will be some confusion over the purposes of learning. Clearly there needs to be some compatibility between the assessment regime and the approach to teaching and learning.

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[1] 'Autonomous' does not mean 'lone'. Groups of learners can act in autonomous ways.

Similarly, autonomy needs to be accepted as well as given. Students who are used to teacher/lecturer direction and have been well trained in finding out what the teacher/lecturer wants and then producing that will need support to take control of their own knowledge building. The concern here, as explained by Broady (1995), is that too much support can lead to dependency while too little can inhibit development.

One established approach to education *in* the profession that provides the possibility for some student autonomy (see, for example, Hatton & Smith, 1995) is case methods—the study, analysis and reflection on particular incidents or examples of teaching (Barnett, 1991). Merseth and Lacey (1993) suggest that case methods can develop skills of critical analysis and problem solving, represent the complexity of practical situations, foster multiple perspectives and levels of analysis, and offer students opportunities to engage directly in their own education. The multimedia resource discussed below is essentially case material presented electronically.

## Using classroom dilemmas in mathematics teacher education

Teachers (including ourselves) face dilemmas constantly: whether to seek to reproduce society or to transform it, to foster specific objectives or broad goals, to present mathematics as universal or culturally determined, to privilege communication or allow it to be free, to develop confidence or to provoke challenge, and so on.

It is appropriate that the word dilemma has two meanings: one refers to an argument with two alternatives, each conclusive against an opponent; the other refers to a difficult choice. In mathematics education the meanings are intertwined—problematic situations lead to the necessity for teachers to be aware of alternative ways of thinking and patterns of behaviour as well as to make informed choices.

The choices are not so much about which poles of the dilemmas to emphasise, or even about determining which poles are compatible with conventional wisdom, but are about how to use the dilemmas themselves to strengthen both curriculum and pedagogy. For example, while mathematics educators have, at times, argued that teachers should pursue relational rather than instrumental learning goals (Skemp, 1976), learning processes rather than specific content outcomes (Newton, 1988), and intrinsic rather than extrinsic motivation (McCombs & Pope, 1994), we suggest that it is better that teachers become conscious of such choices, and be able to use alternative approaches to advantage. Rather than seeing different processes as dichotomous, they can become aware of the continua associated with each dilemma and move along these strategically, according to situational factors operating at particular times (c.f. Berlak & Berlak, 1981). In other words, each pole of a dilemma interplays with its opposite in a productive, dialectical engagement. In this sense, knowledge about teaching is not so much about knowing which choices are correct, but more about having an acute awareness of possibilities and an ability to make informed decisions.

The consideration of dilemmas has become a central principle in our use of technology to provide case study material for education *in* the profession. Electronic technologies have provided us with new tools for making professionals aware of problematic situations in all levels of mathematics education. Our research over the past five years has centred on the development of a computer based resource that facilitates detailed investigation of classroom dilemmas, as is discussed below.

## Using technology to stimulate education *in* teaching

*Learning About Teaching* was a product of a research project that identified the need to use real contexts to facilitate the analysis of teaching (see Mousley, Sullivan & Clements, 1991). Prior to the development of the resource, the project developed a framework for describing elements of quality mathematics teaching. It used analysis of recent literature and a survey of 200 practitioners, teacher educators and other education professionals from several countries (Sullivan & Mousley, 1994a) to identify these components. The results of this research are summarised in Figure 2.

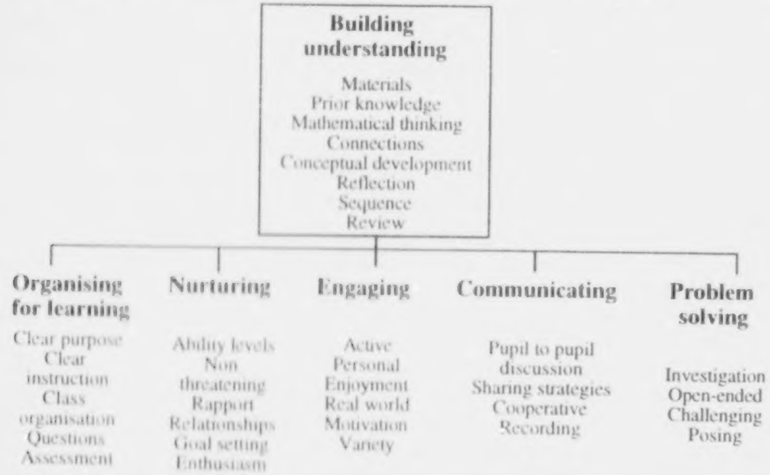
Building understanding
Materials
Prior knowledge
Mathematical thinking
Connections
Conceptual development
Reflection
Sequence
Review
Organising for learning
Clear purpose
Clear instruction
Class organisation
Questions
Assessment
Nurturing
Ability levels
Non threatening
Rapport
Relationships
Goal setting
Enthusiasm
Engaging
Active
Personal
Enjoyment
Real world
Motivation
Variety
Communicating
Pupil to pupil discussion
Sharing strategies
Cooperative
Recording
Problem solving
Investigation
Open-ended
Challenging
Posing

*Figure 2.* Some features of quality mathematics teaching (Sullivan & Mousley, 1994b)

Full lessons that exemplified the components of quality mathematics teaching identified were planned, then taught and videotaped. The videotapes were examined using several techniques, including a qualitative analysis of unstructured reviews of the lessons by over 30 experienced teacher educators (see Mousley, Sullivan, & Gervasoni, 1994). Along with other relevant data and readings, they were transferred to CD-ROM disc. An interactive multimedia computer environment was authored to provide flexible access to all aspects of the lessons and to make links between related aspects of the resource.

The decision to use multimedia was made because of its capacity to show many facets of a professional context. Classrooms are multidimensional and the daily world of the mathematics teacher is a capricious one, but multimedia can give users access to minute features of classroom interaction as well as control over the use of different constructs. We support the claims of Merseth and Lacey (1993) that it has potential for introducing the complexity of pedagogy to novices and experienced teachers alike and that the non linear capability of multimedia allows the use of multiple perspectives and opportunities to review situations.

To capture some of this complexity, the *Learning About Teaching* resource links the videotape of a mathematics lesson to other video records such as pre- and post-lesson interviews with the teachers, procedural documents and readings associated with the lessons, graphic representations of data, a full transcript, etc. This forms an extensive information bank that can be accessed in flexible ways to support detailed analysis of the classroom interaction. Indexing allows scenarios to be examined in conjunction with other data, enabling users to focus on specific teaching skills, moments of interaction, selected sets of incidents, sequences of events, links between theory and the pedagogical action presented, particular students' work, and so on.

The variety of data available for use is perhaps best illustrated by means of an example. Suppose that a group of education students (undergraduate or postgraduate) wishes to use *Learning About Teaching* to explore "patterns of communication" in a mathematics lesson. The participants can choose to examine:

- the teacher outlining her aims for the lesson, including her intention to have the children share their developing understandings,
- patterns of verbal and non-verbal communication across an entire one-hour lesson,
- individual interactions between the teacher and each pupil (as well as with the class and with small groups),
- discussions between children as they investigated a problem,
- questions asked by each pupil of the teacher and of each other,
- a transcript of the lesson (with the ability to cross to the video at any point to check inflection and non-verbal aspects, etc.),
- children reporting back to the class what their groups had done and what they had learned,
- some recent journal articles about types of communication in classrooms,
- some specially written papers about specific aspects of communication (such as types of questions),
- some articles about certain purposes for classroom communication (such as engaging children in higher-level thinking),

- the teacher's responses to particular students' comments and questions,
- graphed data about aspects of communication,
- a map of the classroom (with the ability to click on a group of desks to view the children's interactions),
- the teacher talking after the lesson about how students' contributions were the subject of formative evaluation, and
- a range of other resources.

Besides *Communicating*, the program also facilitates detailed explorations of *Building understanding*, *Engaging learners*, *Nurturing the child*, *Problem solving and investigations*, and *Organising for learning*. However, many other topics could be explored. A recent group of Master of Education students using the resource as a basis for study included the following among their research topics:

- The use of manipulatives to introduce algebraic thinking.
- How did an open-ended question and 'real life' situation enable learners to explore the concept of volume?
- Creating an environment to encourage children to use different modes of communication in expressing mathematical ideas.
- An atmosphere of trust and higher order thinking skills.
- Mathematical constructs: Volume.
- Gender issues in a mathematics classrooms.
- Children's understandings of dimension.
- Seating arrangements and on-task engagement.
- Social constructivism in action.

(Mousley & Sullivan, in press)

The case study of the lesson provides an environment for students to engage in the study of teaching, and the flexibility within the environment creates the possibility of genuine autonomy for the students.

## Using dilemmas as the basis of the study of teaching

A key feature of the resource is the extent to which it raises problematic teaching situations that allow users to become aware that there are dilemmas to be confronted. The basic proposition which underlies the resource is that the study of particular exemplars of quality practice can stimulate reflection on key components of teaching. It is expected that groups of users will investigate and discuss focus questions that aim to draw out varied pedagogical beliefs and to stimulate sensitive responses.

The questions generally focus attention on particular aspects of the lesson where teacher decision-making can be involved, and are intended to provide a stimulus for deeper reflection on problematic aspects of the task of teaching. The following are some examples of focus questions that appear in different sections of the resource:

> *Numbers of questions*
> The teacher asked about 60 questions. Should she take steps to reduce the proportion of her questions and increase the proportion of pupil questions? If so, what steps could be taken?

> *Blackboard*
> In explaining the concept of dimension, the teacher uses a two dimensional diagram on the chalkboard. This has advantages and disadvantages. List these, and suggest some other appropriate strategies.

> *A particular student*
> Some people who write about good teaching practices claim that questions should be directed to quieter and reluctant students. Others argue that if these children are listening attentively such a strategy can be counter-productive. Note Gabrielle's interactions and how she engages with the lesson content. What strategies can be used to encourage full engagement?

> *A mathematical concept*
> Daniel claims there is a fourth dimension. What are your ideas on this concept? Is this an idea that the teacher could have taken further at this stage? Find out how some other people would react to this classroom incident. How could you explore this concept with older children?

The focus questions address a wide range of teaching dilemmas. Users interact both with the resource and each other in ways which can create a sense of control over both the aspects of teaching which are studied and the mode in which they are studied. The presentation of dilemmas in the form of questions linked to various aspects of the resource allows students significant control over the focus of their interactions.

## Articulating one dilemma: Privileging communication

As an example of the way the resource has been structured to raise teaching dilemmas *in* education, let us diverge to explore some of the issues behind just one focus question. We listed above the features of the resource that are useful for investigating *Communication* in the classroom recorded. The resource contains graphs of the numbers of interactions each child had with the teacher, the total duration of these interactions, etc., and clicking on any bar of a graph jumps to the appropriate snippets of classroom video. This allows users to investigate the number of interactions children have in the light of the nature and quality of those interactions. One focus question reads:

> You will notice that the interactions with children were not evenly distributed. What factors contributed to this? If this were a graph of interactions in a lesson you had taken, would you be concerned? Explain your answer. What actions would you take?

This question is essentially about control. In non-institutional social situations, the frequency of people's contributions to discussions and their total time allocation are not controlled. We rarely feel worried about imbalances or attempt to redress unequal participation in everyday conversations. People place themselves within dialogues according to a number of factors, such as their confidence in the situation, their knowledge of the subject matter, the contributions they wish to make (or to reserve), and the roles they wish to play within the group. It is generally accepted that people can be fully involved in discussions by just listening.

However, mathematics lessons are not natural social situations. Traditional patterns of control of communication have developed in schools, just as they have in other social institutions. Like other focus question, this one aims to raise the awareness of a current issue in an active way within the context of the case presented. Thinking is demanded—opinions need to be expressed and justified, alternative actions need to be considered.

Clearly there are competing perspectives. Having all children contribute to classroom discussions has some advantages, and we are all aware of these. For good reasons, we tell our students to distribute questions well, to take affirmative action by including girls or hesitant students, and so on. However, the notions that there are only some acceptable indicators of participation and that teachers should control classroom dialogue arise from a didactic model of education where teachers structure, control and evaluate classroom activity in terms of traditional patterns of interaction. Some other models of education that are beginning to impact on mathematics pedagogy position teachers not at the centre of activity, but as facilitators of a variety of learning processes. Using more natural patterns of interaction, students communicate with each other without continual deference to the teacher and take more responsibility for shaping classroom discourse. Individuals are not put on the spot through discursive imposition. While such models of education do not deny that curriculum content reaches students through the agency of teachers, they do require more flexible patterns of communication within a context of new social relationships and practices in classrooms—and do require traditional teacher actions to be challenged.

Exploring the issue of patterns of oral participation within competing views of education requires examination of how and why learning processes are limited in particular ways. As Walkerdine (1990) notes,

> ... the issue of silence and speaking is not a simple matter of presence or absence, or of suppression versus enabling ... what is important is not simply whether one is or is not allowed to speak, since speaking is about saying something. In this sense, what can be spoken, how, and in what circumstance is important. It tells us not only about its obverse, what is left out, but also directs attention to how particular forms of language, supporting particular notions of truth, come to be produced. This provides a framework for examining how speaking and silence and the production of language itself become objects of regulation. (p. 54)

Thus this particular focus question is about how linguistic patterns shape, constrain or facilitate social relations—of mathematics teachers with students, students with each other, and students with curriculum content and tools.

### From dilemmas to action

Dilemmas are, of course, merely a subset of the decisions and thinking in which teachers engage. Figure 3 represents an inclusive hierarchy of the relationship between dilemmas and overall teacher thinking.

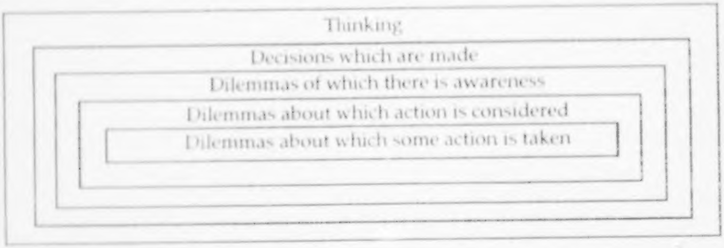
Thinking
Decisions which are made
Dilemmas of which there is awareness
Dilemmas about which action is considered
Dilemmas about which some action is taken

*Figure 3:* Relationships between dilemmas and thinking

A perturbation of pedagogical schemata seems to provide some salience, and so has potential to create opportunities for individuals to see the need for change. However, pedagogical schemata are persistent, deep-rooted and well-organised structures that shape both thinking and behaviour—perturbation is merely a first step. Providing opportunities to explore pedagogical dilemmas in the company of peers (as opposed to authorities) can also do much to raise awareness of alternative beliefs and actions, to help teachers make informed and defensible choices, and to bring new schemata into productive relationships with established ones.

Nias (1987) points out that for new perceptions and beliefs to be internalised they need to be developed with peers, and not in authority-dependent contexts. She claims that

> The greater (the schemata's) association with feelings of dependency, the more difficult it will be for an individual to raise them to a consciousness level, to own or to change them. Without this kind of 'consciousness-raising' and in the absence of alternative schemata, teachers will continue to make the same judgements. Whatever else may alter in their school situations, their teaching—dependent as it is on stable patterns of perception—is likely to continue unchanged. (pp. 5-6)

The broader goal of mathematics teacher education, however, must be to support teachers' actual change in classroom actions. Figure 4 indicates change options.

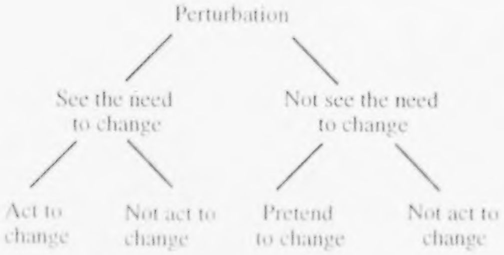

*Figure 4:* Processes of stimulating change

It is hoped that using the *Learning About Teaching* resource will assist our students to be aware of alternative ways of thinking and patterns of behaviour, as well as to make informed choices—that is, to make a start in a movement along the left-hand path.

## Using the resource to provoke reflection and discussion

In 1995, one group of students who used *Learning About Teaching* were second year undergraduates—pre-service teachers. Interaction with the resource fairly autonomous, in small groups, for approximately 20 hours. The aims of this activity included close analysis and discussion of the lesson, linking of theory and practice, and stimulating learning about practice through reflection on practice.

Data on students' use of (and learning from) the resource were sought through structured survey items, the writing of lesson critiques, responses to formulated scenarios, unstructured interview items, observation of students' interactions with each other and with the program as they worked, journals maintained throughout the process, and reflective essays written after the completion of the program. We were particularly interested to find out:

- how the case study was used;
- whether this use of technology promoted improved observational skills and understandings of processes involved in teaching and learning mathematics;
- whether it promoted reflection, discussion and writing by groups of teacher education students; and
- whether students became actively involved in the consideration of teaching dilemmas and implications for classroom action.

Space allows only a few snippets of data and a summary of our conclusions. The following is the response of one user, Karrie. This student was selected by research associates who were asked to choose a case that was representative of the data overall and which conveyed a sense of the impact of the resource on individual users.

Prior to the use of the resource, users were asked to write a short open response to describe what they saw as the features of quality teaching. Karrie wrote:

> Clear examples completed together as a class. Clear instructions. Clear formulas - shown step by step. Clear diagrams. Sufficient data - easy for group to interpret.

To the corresponding question asked after the use of the resource, Karrie responded:

> Children be able to think for themselves. Every individual catered for. The teacher knows who's going to be the 'strugglers' and who's going to be the 'good' ones so to speak. Aware of likely problem areas. Involve everyone in active learning.

Prior to using the resource, when Karrie was asked what she would expect to see in a quality mathematics lesson, her response suggested a relatively narrow view of teaching:

> Good communication, between teacher and students. Ah, clear description of tasks. Clear, sort of, instructions. Clear diagrams so that the children can understand them. Ah, clear formulas, that are easy to understand ... And just going around to help them, the students.

After using the resource, she demonstrated a keener interest in the teachers' role in relation to developing children's thinking:

> The teacher goes around the class while they are discussing, listening, probing. She alternates who she asks questions. Clear step by step instructions. Uses blackboard to demonstrate ... Sets the children to work well. Uses a practical application for the task. Constant interaction with the children.

Admittedly the changes in the responses are subtle. It is hard to gauge from one example how typical this shift to a focus on teacher actions was, yet it was clear that the students made overall progress in their understanding of the many different roles that teachers can play in mathematics classrooms. The research associates believed that Karrie's responses were fairly typical. They wrote:

> Initially, Karrie gave the impression of seeing teaching as telling the students what to do as clearly as possible, being sensitive to the needs of individual students in an affective sense, and of believing that teaching should be made fun. After the use of the resource, as well as developing a clearer language for describing teaching, Karrie articulated a desire to emphasise the thinking of the children, and catering for the particular needs of individual students.

Our summary of changes in the responses of students made after analysis of the data also demonstrated progress. A snippet from our notes reads:

> Karrie expanded her vocabulary for talking about teaching, has more insights into teaching processes and was more thoughtful after using the resource. There was a change from a view of teaching as instructing to teaching as a process of engaging children. The focus on the students, and particularly sensitivity to their affective needs, which was evident beforehand was still apparent ... On balance it seemed that Karrie developed some new understandings about quality teaching, extended her ability to discuss the finer points of classroom interaction, and became more skilled at classroom observation and analysis.

We also sought data on the way that students responded to this alternate mode of learning. It was clear from many students' written comments that the discussion of focus questions was a productive part of the experience. For example, an entry in Karrie's journal read:

> This set of focus questions took the longest so far. The reason was because none of us seemed to agree on any of the answers. We constantly needed to refer to earlier work and the readings ... We all had different perspectives, but could see advantages of other people's suggestions as well as our own.

Not only is there a suggestion that the dilemmas are being considered and even debated, there is reference to using the whole resource to support particular arguments, as well as acknowledgement of the legitimacy of competing perspectives. This can be taken to be indicative of a sense of ownership of the learning experience.

It also seemed that Karrie appreciated the relatively autonomous mode of learning—both its flexible nature and also the group discussions. After the use of the resource, Karrie wrote in her journal:

> By using the multimedia program, it enabled me to work at my own pace. It provided a concrete example to analyse and interpret. Also working in groups provided an opportunity to hear and discuss other interpretations and experiences. The program enabled us to bring our own experiences in to it and compare and contrast through discussion ... Being able to work at our own pace was a bonus in learning, as it enabled us to go back over anything we didn't understand and watch it again. This is something that just isn't possible when studying in lectures and tutorials. Also working in groups added insight as there was a feeling that we weren't sure if we were right or wrong with our responses to the questions ... I think this is a key in learning, experiencing for ourselves and learning from others.

This seems to be a clear endorsement of both the sense of autonomy created and the value of learning through discussion and experience.

In summary, students' unprompted journal entries frequently indicated that the flexibility offered by the mode of learning was appreciated, and that the resource provided rich source material for intense, thought-provoking discussions and investigations.

## Conclusion

Nias (1987) notes that pedagogical schemata become integrated into consistent wholes so that we respond as much to distant patterns or verbal cues that suggest them as we do to specific stimuli. Electronic technologies are proving useful tools which enable these wholes to be brought to a level of consciousness and hence made susceptible to analysis and critique. Individual tools such as audiotape and videotape have been used successfully in the examination of mathematics pedagogy, but multimedia has the added capability of assisting users to draw active links between

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different media, allowing the multidimensionality of teaching and learning to be represented and explored.

Using a broad-based multimedia resource has given our mathematics education students opportunities to undertake relatively autonomous investigations. The case material has enabled learning *in* education as well as *about* education. Not only the fullness of a real classroom has been brought to the study of teaching, but also the fullness of the experiences of the learners. The practical examples provided as case data can link to, and even reinforce, the experience of groups of users, but their interaction provides opportunities to explore alternate perspectives on the lesson text.

Our use of multimedia aims to develop teacher's understandings of the complexities of mathematics classrooms and the roles that teachers play. Our research on the potential of using pedagogical dilemmas to make undergraduate and postgraduate mathematics education students more aware of the existence of alternative pedagogical perspectives is not complete, but has already provided some promising data.

## References

Anderson, J., & Alagumalai, S. (1996). *From text-based pedagogy to cyber based methodology.* Paper presented at the Educational Research Association and the Australian Association for Research in Education joint conference, Singapore, November 23-28.

Barnett, C. (1991). *Case methods: A promising vehicle for expanding the pedagogical knowledge base in mathematics.* Paper presented at the annual meeting of the American Educational Research Association, Chicago, April 3-7.

Bednarz, N., Gattuso, L., & Mary, C. (1996). Changes in student teacher views of the mathematics teaching /learning process at the secondary school level. In L. Puig & A. Gutierrez (eds.), *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 59-67). Valencia, Spain: Universitat de Valencia.

Berlak, A., & Berlak, H. (1981). *Dilemmas of schooling: Teaching and social change.* New York: Methuen.

Broady, E. (1995). You are your own best resource: Promoting confidence in autonomous learning in teacher education at a distance – A case study. In R. Howard & I. McGrath (eds.), *Distance education for language teachers* (pp. 49-61). Adelaide: Multilingual Matters.

Bruce, N. (1995). Practising what we preach: Creating the conditions for student autonomy. *Hong Kong papers in linguistics and language teaching, 18,* 73-88.

Chazam, D., ben Chaim, D. & Gormas, J. (1996). Overcoming isolation in teaching: A case study of collaboration in a professional development school mathematics department. In L. Puig & A. Gutierrez (eds.). *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 217-225). Valencia, Spain: Universitat de Valencia.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp????). New York: Macmillan.

Corsine, R. J. (1987). *Concise encyclopedia of psychology.* New York: Wiley & Son.

Guskey, T. R. (1986). Staff development and the process of change. *Educational Researcher, 15*(5), 5-13.

Hatton, N., & Smith, D. (1995). Reflection in teacher education: Towards definition and implementation. *Teaching & Teacher Education, 11*(1), 33-49.

Levenberg, I., & Sfard, A. (1996). When change becomes the name of the game: Mathematics teachers in transition to a new learning environment. In L. Puig & A. Gutierrez (eds.) , Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education (vol. 3, pp. 249-257), Valencia, Spain: Universitat de Valencia.

McCombs, B.L., & Pope, J.E. (1994). *Motivating hard to reach students.* Washington: American Psychological Association.

Merseth, K. K., & Lacey, C. A. (1993). Weaving stronger fabric: The pedagogical promise of hypermedia and case methods in teacher education. *Teacher and Teacher Education, 9*(3), 283-299.

Morgan, A. R. (1996). *Still seeking the silent revolution? Research, theory and practice in open and distance education.* Paper presented at the Research in Distance Education conference. Geelong: Deakin University.

Mousley, J., & Sullivan, P. (in press). Using multimedia in context-based pedagogy for distance education students. In T. Evans & D. Nation (eds). *Research in Distance Education 4.* Geelong: Deakin University.

Mousley, J., Sullivan, P., & Clements, M.A. (August, 1991). *The perceptions which student teachers have of teaching practices in classrooms observed during field experience.* Paper presented at the thirteenth annual conference of the Mathematics Education Lecturers Association, Perth.

Mousley, J., Sullivan, P., & Gervasoni, A. (1994). The analysis of teaching: Constraints in lesson description and critique. In G. Bell, B. Wright, N. Leeson, & J. Geake (eds.).*Challenges in mathematics education: Constraints on construction* (pp. 437-444). Lismore: Mathematics Education Research Group of Australasia.

Newton, W. (1988). Graves, process maths, and the need for a theory to guide the teaching of mathematics. *Prime Number, 3* (2) 9-10.

Nias, J. (1987). *Seeing anew: Teachers' theories of action.* Geelong: Deakin University.

Peluso, T., Becker, J., & Pence, B. J. (1996). Extending the educational conversation: Administrators view of staff development. In L. Puig & A. Gutierrez (eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (vol. 4, pp. 115-122), Valencia, Spain: Universitat de Valencia.

Queensland Board of Teacher Registration (1993). *Environmental education: An agenda for pre-service teacher education in Queensland.* Brisbane: QBTR.

Schon, D. (1983). *The reflective practitioner.* New York: Basic Books.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Sullivan, P., & Mousley, J. (1994a). Quality mathematics teaching: Describing some key components. *Mathematics Education Research Journal, 6*(1), 4-22.

Sullivan, P., & Mousley, J. (1994b). Making classroom communication natural. In C. Beesey & D. Rasmussen (Eds.) Mathematics without limits (pp. 428-434). Melbourne: MAV.

Walkerdine, V. (1990). On the regulation of speaking and silence: Subjectivity, class and gender in contemporary schooling. In V. Walkerdine (ed.), *School fictions* (pp. 29-60). London: Virgo.

# OPEN TOOLSETS: NEW ENDS AND NEW MEANS IN LEARNING MATHEMATICS AND SCIENCE WITH COMPUTERS*

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**Abstract:** *Open toolsets are a new genre of software that involves a greater number of smaller units than conventional educational "applications." The units are intended to be highly modifiable, extendable, and combinable with each other. Primarily with examples created in the computational medium, Boxer, I illustrate open toolsets. I suggest how they can be constructed and what learning properties they may have. In particular, I argue that open toolsets may allow very different development communities, involving teachers in essential ways. Finally, I outline areas of research that need to be done to support the best development and use of open toolsets.*

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## Introduction

In this paper, I wish to introduce, explain and advocate a new genre of educational software. My interests are mainly in mathematics and science education, but I am quite sure what I say here will travel well into other disciplines. After introducing the notion of "open toolsets" and the standards by which they should be judged, I will spend a significant amount of time on some technical issues. This is because technology sometimes really makes a difference in what we can accomplish educationally, and it happens that this new genre of software is only just now becoming technically feasible.

*Open toolsets* are, at first blush, just what the name implies. They are an open collection of tool-styled software units that are aimed toward learning in some particular subdomain, like constructions in geometry, system dynamics, particular pieces of ecology or evolutionary theory. "Open" is meant in several senses. First, each tool in the set has the usual properties of a tool. That is, it helps accomplish tasks and is pedagogical to the extent that those tasks are educationally relevant. Tools, in general, trade off explicit representation of educational goals, which may be more evident in other, more didactic software, for a sort of authenticity in which software is instrumental to the goals of its users.

My motivations for developing tool-styled software are both aesthetic and instrumental. Aesthetically, I am committed to maintaining a sense of personal agency and responsibility in the users of educational software, teachers and students alike. This aesthetic goal also has instrumental overtones. If we want responsible, self-motivated learners and thinkers, we need to set up learning environments where those orientations are cultivated. But, I also persist in reminding that the educational experiences of our students are not just steps toward practical ends—for example, to get and keep good jobs—but they are also an expression of the values of our civilization.

We cannot escape deciding how we want our offspring to experience life in school, beyond any practical measures of utility. I want them to feel directly empowered.

The more direct educationally instrumental value of tools, for me, follows from a basic orientation toward knowledge—that it is cultivated best in the service of goals understood by learners. I won't explain this presumption in detail here, except to note two things. Working within learners' goal systems serves "motivational" ends, keeping them engaged. But this also has epistemological ends. Goals help refine knowledge by providing an understood "measuring stick" to determine if current knowledge is powerful enough. Understood goals have high epistemological leverage.

So far what I have said needs only the "tools" part of the open toolsets moniker. The "sets" adds an additional dimension. These days, I believe software generally comes in too few and too big chunks. Flexibility is a key issue. The primary way to achieve flexibility in a big software system is by adding an endless stream of features. I am not the first person to note the relentless commercial pursuit of features and the downside for consumers. Mike Eisenberg (1995), for example, writes eloquently about this issue. Elegance, comprehensibility and mastery generally just fade away when software systems become too complex. Although a full-time user may appreciate a huge range of specialized features, when software is used educationally, we frequently pass from one piece to another as our specialized needs for learning evolve. So, in general, students and teachers just don't have time or need for exotic features hidden in a maze of options, subtools, preferences, and the like. The route advocated here is many simpler, easier to understand tools, compared with fewer, more complex ones.

There is a place for complex, broadly useful tools. I have been a long-term advocate of a very rich set of generic capabilities, like text and hypertext processing, graphical, network and programming facilities as the basis of students' experience with computers. This is the very concept of "computational medium" that has motivated my own work with computers in education for a decade and a half (diSessa, 1995). But the concept of a computational medium suffers the same "problem" as generic literacy. Knowing how to read is only useful if there is a literature about the subjects you want to know something about. (And, incidentally, learning to read is best accomplished in the context of reading about something you care about.) So, open toolsets is a candidate for the content "literature" that will be the basis of learning particular things with computational media. These toolsets will interpolate between the best general-purpose environment we can imagine, and the scores or hundreds of learning micro-contexts we must create for students to become mathematically or scientifically literate.

We've gotten as far as "tools" and "toolsets." But why the apparent redundancy of "*open* toolsets," if the very meaning of tool connotes a degree of openness? The answer is that I want to emphasize toolsets that will never be finished and complete, but will always be open to changing old tools and adding new ones. Changing old tools is the easier of these desiderata to grasp. Tools like hammers and saws are flexible in one way—they don't "care" what they are used for. But they are not in themselves malleable and changeable. With electronic tools, if we are clever, we need never be limited in this way. Although, again, this is not a trivial cleverness. Almost all current software tools are much less changeable than I believe optimal for open toolsets.

The second part, adding new tools, is simple enough in principle. But there is still another level of openness. The key to this other level is that open toolsets work best with an important degree of synergy among components in the set. Each tool needs to be open to working with an unlimited range of new tools that might be added to the set. Merely having the toolset running on the same machine is far from sufficient. Having a toolset with components that can work flexibly together is critical, but still insufficient. Instead, the ultimate principle of synergy is that tools need to be interconnectible and combinable in ways that are not anticipated by the initial toolset builders. Indeed, they need to be combinable with other tools that may be added to the set in the future.

I would like to begin to articulate the criteria by which the success of open toolsets may be measured in more definite terms. To be sure, such software must support good learning by students in classrooms. As I said before, this learning will be critically dependent on the activities that we foster among students and teachers. There is quite a lot to think about with regard to this, and at least as much to design. But my emphasis here will be in a different direction. The idea of open toolsets is motivated in substantial degree by issues of the appropriation of technology deeply into existing cultures, and the co-development of technologies and new cultures.

I like to think about technology fitting and developing "social niches." These are the broad, repeatable patterns of production and consumption defined by the multiple social constraints of value, capability and interest that impinge on genres of technology and their use. The simple idea is that any software type, open toolsets in particular, must suit many people's interests and ways of working. These pose strong constraints on what types of software exist and can come to exist.

Let me illustrate social niches by talking about what I propose for a new mode of software development for open toolsets. Currently almost all software development is done by teams of experts. Teachers and students, if they are involved at all, are testers and commentators. The paradigm I have most hope for with open toolsets is that these will be developed in larger groups, which include teachers and students doing important work in the design. Over an extended period of time, tools are tested and modified, activities are designed and shared, as a community with solid ties to the educational practice converges on a toolset that is both powerful and flexible enough to serve a wide range of local styles and interests. There may well be software developers (programmers) in the community, but far more time will go into building a joint culture around the toolset than into "coding" per se. We need to reverse the emphasis on code and interface and put it more appropriately on activities and cultures.

Is this a viable image? It certainly has attractive properties. But there are important uncertainties. First, how do people make money out of this scheme? The first law of capitalism is that, unless someone expects to make a lot of money, the whole program is unlikely to succeed. There are some scenarios that lessen this threshold. For example, if open toolsets prove extremely valuable in some experimental context, government in some form may decide it is in the public interest to facilitate their development.

A second critical issue is the expertise of teachers. Are there enough creative and thoughtful teachers for that subculture to take a substantial role in "software development," even if it is a very different form of development and a different form of software at issue? I am more confident than most people that there is a tappable, or at least developable subculture of teachers who can lend an important realism and "bottom up" sensibility to software development by contributing to open toolset creation. Even if this works within development communities, will the miniature communities that build toolsets be representative enough that other teachers, not in the development community, can easily join in? These and a host of other complex and uncertain issues about the viability of this paradigm are basically asking, "can we foster an effective social niche for open toolsets?"

Without proposing to answer such questions—we just may not know the answers without trying—let me review the rationale for open toolsets. First, an open toolset is a flexible collection of smaller, less "expensive" to build parts. We want to preserve initiative and ownership for students and teachers by designing a flexible base for learning and instruction, a base that grows and can continue to grow organically without ever imposing straightjacket-like constraints. Students will do "work" they understand and with which they can personally identify using these tools. Teachers, also, will be served by the toolset. They can exercise their personal style, teaching sensibilities and sense for their own students and local context in designing and modifying activities with the tools, and even modifying the tools themselves (and combinations of tools) involved in activities. These properties can help with the critical task of teachers' appropriating technology into their professional practice. Open tools "meet teachers half way" in giving them more control over the technology. This extends further the fact that open toolsets may be developed in substantially different ways than current software; the school community may have a much more substantial *and sustained* role than at present.

No new idea is completely new. Open toolsets are an extension of many current ideas about learning with technology. They are in the same spirit as "constructionist" and microworld approaches to learning, popularized, in particular, by Papert and the MIT Media Lab (Epistemology & Learning Group, 1991). I also already mentioned Eisenberg, at the University of Colorado, in connection with his attempt to combat complex but still inflexible applications via programmability. Another project that is working toward many of the open toolset goals is the SimCalc project (Roschelle & Kaput, 1996).

## Technology

Technically, how do we create these open toolsets? One obvious possibility is that each tool is a separate application. This is certainly compatible with the general framework of current computational systems. But it poses a range of limitations and problems that have become the subject of a lot of discussion and attempted "fixes." First of all, applications tend to be large and complex. Because the operating systems currently available supply only a limited range of services to users, applications must build from scratch, and they frequently duplicate each other's basic services. A good example is text editing. Almost every application does or can make use of text processing, but anything beyond the most basic services are built anew for each application. This is not only technologically inefficient, but leads to difficulties in learning—each implementation of a service will have idiosyncracies that users must learn in moving from application to application.

Other limitations of using separate applications are weaknesses in sharing data and in interaction. Each application typically has its own data types. You are lucky to be able to cut and paste text (and usually you will lose formatting) and simple pictures, much less a complete interactive object. Why shouldn't a business document contain an active spreadsheet? Or why shouldn't a student's science report contain both the data she used and the analysis tools, so the teacher can check the analyses or run a different one to show, for example, how a different conclusion is also supported? In terms of interaction, it is currently very difficult to connect different tools-as-applications. Suppose you want to connect a nice simulation, say of an ecological system, with a statistical analysis tool and a graphing tool. This is sometimes possible these days, but as often it is awkward, and too often it is just plain impossible.

There are three movements in contemporary computing that are related to each other and to the problems described above. First, "scripting" is really just the ability to control an application (or a component—see below) with a programming language. This means applications can at least use each other's resources. You could, for example, have a script that sent some data to an analysis tool, and returned a graph-as-picture to your "home" environment (say, a text editor). You can essentially create a new application by gluing together resources provided by multiple existing applications. Clearly, this is in the spirit of open toolsets, and it is an important step toward realizing their promise.

"Component computing" is an attempt to allow multiple, small "applications" to work together efficiently. Microsoft established one standard, OLE (object linking and embedding) with a follow-up called ActiveX. Apple and IBM pursued another standard (OpenDoc, now apparently defunct). A newer possibility might be most familiar to you. The programming language Java, which is most visible in network applications, has a standard that allows multiple little "applets" on the same page of a net browser. Each job you want to do might require a different collection of several small applets, rather than several big applications. "Document-centered computing" is a slightly different way to describe component computing. The idea is you should always work on the basis of the document you want to create. The resources you need for the different parts of the document (text processor, picture editor, analysis tool, spreadsheet...) should just come with the document. You should not need to run off to load a big, separate application just to work on a piece of your document.

In this paper, I will use a rather different technological basis for making open toolsets feasible. I will use Boxer, the system we have been designing, implementing and testing for quite some time at the University of California, Berkeley (diSessa, Abelson, & Ploger, 1991). Boxer has some advantages over the movements described above. In particular, it makes full programmability by a simple programming language (simpler than Logo for beginners, we believe) the heart of its capabilities. But it also has disadvantages. The most obvious disadvantage is it is not "industry standard," and therefore it is not ubiquitous. I won't continue the discussion of advantages and disadvantages of Boxer here, even though these are interesting and important. That is because the point of this paper is to paint a picture of what open toolsets might be like, and what they would achieve educationally.

Before we can begin serious assessment of the educational possibilities of open toolsets, I want to give a more detailed sense of what such tools might be like. I will do this by describing a number of examples, implemented in Boxer. For more technically oriented readers, I note that the capabilities that make these examples possible come in three levels. First, any component system allows a level of combinability. Second, some possibilities described below also require programmability. Finally, a few require a computational medium, like Boxer, where every object is computational.

### A Visible Calculator

Let me begin with a trivial example—a simple calculator. I do this mainly because everyone knows what calculators are and how to work them. So, what is new in this example is exactly what makes this an open tool, unlike the calculator you have in your desk drawer, or more appropriately, the one on your computer. You should be able to see clearly just exactly what "opening" this sort of tool can mean.

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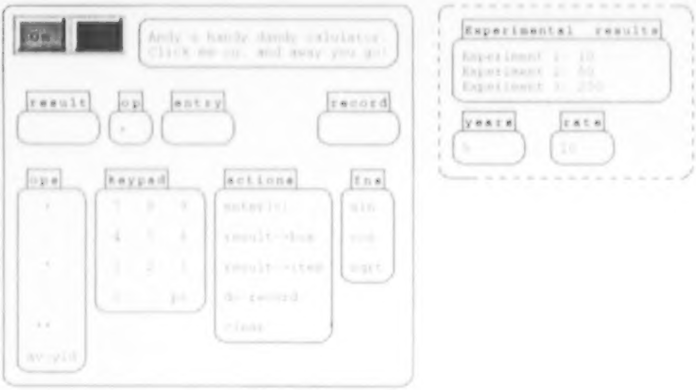

Figure 1. A "visible calculator" (left) and some of the surrounding environment (right).

Figure 1 shows a Boxer "open calculator" (the box on the left) and a few pieces of the surrounding environment (on the right). This is pretty much generic Boxer. Boxer is an environment constructed out of boxes that contain text and pictures. Text and pictures, themselves, can contain more boxes, and hence more text and pictures, in meaningful box-chunks. Here, the calculator is a box. It contains, for example, a keypad, which also is a box, and on and off buttons, which are boxes that happen to have a graphical "boxtop" presentation (icon). You can "flip" the graphics to see the *real* box underneath. Each of these, the calculator itself and its parts, might be considered components in a component computing model. In particular, in Boxer you can cut, copy and paste any box (and text and pictures, of course, also). This means you can move your calculator into any workspace or document you like within Boxer. You can also paste a copy of the calculator in a document to leave as a permanent part if it, for example, to allow users of the document to do their own calculations with the data provided. Why shouldn't a tax form come with a calculator, or a written assignment for a student come with the tools s/he needs to do the assignment? Why shouldn't you be trying out the calculator in Figure 1, rather than just looking at it?

- *Open tools can be cut and pasted into any document or work environment, and left there for any future work.*

Boxer adds a new and important sense to "open" in the tools it allows. The boxes that serve as display devices, **`result`**, **`op`** (operations) and **`entry`**, show you the actual working state of the calculator. They are, in fact, simply variables in Boxer, which are nothing more or less than named boxes. You are literally looking at a part of the working mechanism of the calculator, not just a display. In this case, I (as designer) decided it might be more useful to see the two numbers you are going to operate on and the operation that is going to be performed, in contrast to just the standard single number display. This form of openness, showing the works of the tool, is pretty simple in its implications here, but not completely trivial. For example, you can see what is going to happen when you push the **`enter(=)`** key; you don't have to guess. Did you accidentally push `/` rather than `+`? What happens if you press `123 + 54`, and then decide you meant `123 - 54`? Do you have to start over, or can you just press `-`, then

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**`enter(=)`**? (It happens that you can just do the latter, which is nice.) What is the difference between pressing **`enter(=)`** the key once and twice? Since you can see the "internal" state of the calculator, you know what happened and what is going to happen. When tools get more complicated than a calculator, being able to easily show some of the internal state and workings is even more of an advantage, as later examples will better illustrate.

- *Open tools can be open in the sense of "transparent." They can show users what is going on.*

The calculator has an important property, trivial modifiability. The keypad is nothing more than some text typed in a box. So, if you happen to like a different arrangement of keys, just do it! In fact, I added the **`pi`** to the calculator since I frequently do scientific calculations that need it. To add it, I positioned my typing cursor and typed "p" i". Not a difficult task. And if you don't want that "key" in the keypad, just select and cut.

- *Some aspects of open tools may be trivially modifiable. Accomplishing a change in the tool may be almost as easy as thinking of it.*

Of course, modifying **`ops`** (operations) and **`fns`** (functions) may be just as simple. Sin (sine), cos (cosine) happen to be built into Boxer, so inserting them was as easy as typing three characters. Similarly, a log function or tan function come and go from the tool with a few keystrokes. Modifiability and adaptability are hallmarks of open tools.

Now, trivial modifiability just may not do the job. Not every function or special number is built into Boxer. Suppose you need a statistical function, like standard deviation. Suppose you need a financial function like average-yield, in order to compute the average yield of an investment that appreciated 50% in 5 years. (Note in Figure 1 that **`av-yield`** actually appears in this calculator!). The calculator was written so that any function that you can write in the Boxer programming language can be added to the calculator by cutting and pasting the function into the calculator works, and, again, adding its name to the appropriate keypad. In this case, **`av-yield`** has been pasted into a hidden part of the calculator, its closet. Every box has a closet, so this is something most Boxer users, especially tool users, know how to find and use.

- *Boxer open tools are always open to "extensive modification" as well as trivial modification. You can program extensions or changes in how the tool works.*

If a tool is well-written and you understand Boxer programming, "extensive modifiability" may be nearly as simple as trivial modifiability. This means a lot for Boxer literate folks. If means they can't be straighjacketed by any supplied tool. But it means just as much in a community that includes Boxer novices. If a novice makes a suggestion, frequently a more expert colleague can just make the change on the spot.

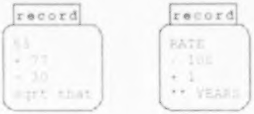
We haven't even gotten to the more interesting aspects of this calculator.

- *Open tools interact well with their environment. They are an integral part of a flexible workspace.*

It happens that you can click on any number outside of the keypad to enter something to compute with. This may seem trivial. It may just save a few keystrokes. But fluidity is not to be underestimated. If you take half the drudgery out of a job, a lot more learning can occur. And, this same principle, excellent integration of a tool with its environment, has deeper and more important forms than saving keystrokes. Examples to come will illustrate this.

Getting data into the calculator suggests that one might want easy ways of putting data out. The actions **result->item** and **result->box** allow you, respectively, to replace any number outside your calculator with the result, or to append the result to a box. So, the calculator has as many external memory registers as you like, and those can be within explanatory text (**Experiment 1**, ... in Figure 1), if you so choose.

- *Boxer Open tools are computational in several senses. Sense (a): Since Boxer has (is) a programming language, it is exceedingly easy to capture, reflect on, and change a process.*



```
55
+ 77
+ 30
sqrt that
```

```
RATE
/ 100
+ 1
** YEARS
```

Figure 2. A record of a simple calculation (left), and one using variables (right).

The record box in the calculator automatically saves a record of the calculation you perform. This turns out to be terribly easy to program. Every operation on the calculator, a mouse-click here or there, winds up executing a Boxer command. The record is just a slightly modified listing of these commands. Many Boxer tools and microworlds benefit from such recording capability, which might be called "macro construction" in some systems. Not only can you review your work, checking for mistakes, but you can edit the record and re-run it (action **do-record**). Figure 2 shows two such records. The first is quite ordinary: adding up a series of numbers and taking the square root. (**That** is a convenient notation for the result of the previous calculation. It is handy to use with prefix operations.) The second record uses a special computational feature: Instead of clicking on a number, we clicked on the name of the variables **rate** and **years**, which therefore appear in the record. This still computes the "formula" entered, which is, in this case, the result of compounding a certain rate of interest for a certain number of years. But, since **record** contains symbolic values, you can easily edit **rate** and **years**, and then **do-record** will compute a new result. Again, this really entails almost no extra learning: Any user of Boxer knows how to name a box to create a variable. And it requires almost no effort for the developer: Inside the calculator mechanism, within the Boxer code that runs it, a variable is always as good as a explicit number.

- *Computational sense (b): Boxer open tools can make use of a wide range of computational structures to make them both more flexible and easier to construct. For example, the written representation of a process can always be abstracted by using symbolic names, variables, in place of explicit values.*

Since computational sense (b) may be unfamiliar, let me further exemplified. This calculator politely explains obscure errors by replacing the introductory text "Andy's handy dandy..." with a message. Many Boxer tools are similarly informative.

- *Boxer open tools are often self-documenting; they can "talk to you."*

Of course, this idea is not special to Boxer tools, but could be implemented within any tool or component. However, Boxer makes this sort of thing particularly easy. The introductory text happens to be just another (unnamed) variable that is set by the calculator mechanism according to its internal state. (Boxer allows unnamed variables by using a special structure called a port. You may use a port to a box in any place you might have used the name of the box.) The generalization of this fact is critical for the open toolset program. For moderately expert Boxer programmers, creating a tool like this calculator is very easy, given the wide range of simple but useful computational capabilities built in to Boxer.

Here is yet another example of computational sense (b). The part of the calculator that re-runs the **`record`** is essentially a one-line program that sequentially executes each row of the **`record`** box as if it were a programming statement. The whole calculator was about a day's work for me, and most of that was design rather than coding. "A larger number of simpler tools" is well within reach if each tool takes a day or so to create. And, once again, we can turn our efforts to activity design and improving the tools through extensive use rather than "getting it right the first time."

Let me begin showing some tools whose pedagogical use is more novel than the calculator's.

### A Simple Graphing Tool

Figure 3 shows another Boxer open tool. It is a simple graphing utility designed mainly to show time-parameterized data. Again, this was essentially a one-day design and programming project. The basic operations of the grapher are controlled by a pulldown menu that appears when you press your mouse button anywhere on the graphing tool.

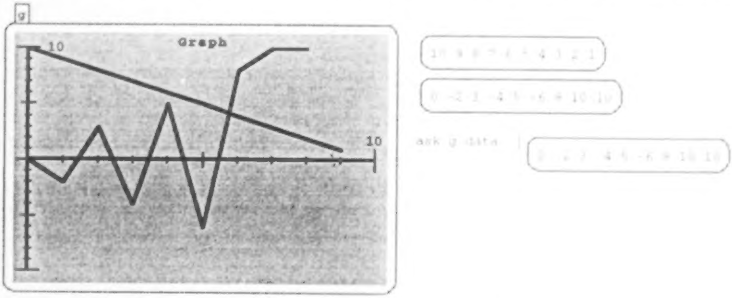


Figure 3. A graphing tool (left) and some of the surrounding environment (right).

Integration with the work environment in this case includes several capabilities quite similar to the calculator. In particular (after using the pull down menu to select **`graph-data`**), you can just click on a box full of numbers to graph them. In Figure 3, the top two boxes to the right of the graphing tool were graphed, changing the color of the graph in between. A similar capability is that one can direct the graphing tool to create a graph in real time, as a simulator runs or as some data arrives from an external probe. In this case, the ability to script the graphing tool is important. You can just "**`ask g plot a-point`**", where **`a-point`** is whatever datum you want to plot. After you collect a set of data in a graph, point by point, you can also "**`ask g data`**," which results in handing you the set of values last plotted (see the third item in the column to the right of the calculator in Figure 3). Of course, you can cut, copy and paste the graphing tool wherever you wish. In case you don't want a copy of the full tool, but just the picture it shows, you can use another generic Boxer capability. **`Snap g`** (**`snap`** as in "snapshot") returns a copy of just the graphics of any tool like this one.

"Recording a process," like the calculator's **`record`** feature, takes the form of a

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pulldown menu option to draw a graph with the mouse. **Ask g data** then fetches the numerical data that you entered graphically.

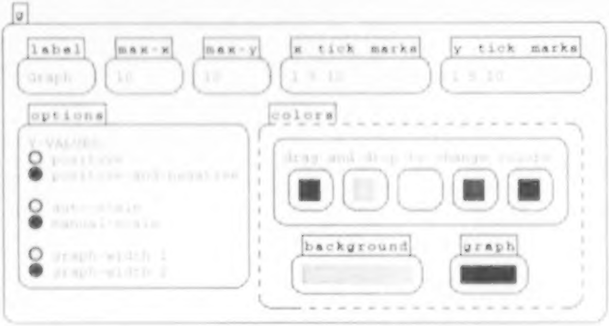
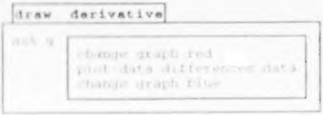

g
label: Graph
max-x: 10
max-y: 10
x tick marks: 1 5 10
y tick marks: 1 5 10
options:
Y-VALUES:
○ positive
● positive-and-negative
○ auto-scale
● manual-scale
○ graph-width 1
● graph-width 2

Figure 4. "Flipping" the graphing tool box shows its settings.

Figure 4 shows the built-in options that you can adjust for producing graphs. As I mentioned before, you can "flip" any graphics box to see its "real" box contents, and these options are on the flip side of the graphing tool box. You can adjust the label of the graph, the maximum x and y values, the places where tick marks are shown on the axes, whether the graph plots negative as well as positive values, whether the scale (maximum y) is automatically computed for a given set of data, and the width and color of the graph lines, as well as the background color of the graph. In showing these options, my real motivation is to show another aspect of an open toolset. Tool building is recursive! These options were created by cutting and pasting other existing tools into the graphing tool. In particular, the little subtool that allows you to drag and drop new colors onto the color variables was taken directly from a general tool library we are accumulating. Similarly, the "radio buttons" that allow selecting some options were taken from the same library. Finally, the pulldown menu that controls the graphing tool was also taken directly from that library (as were the on and off buttons in the calculator).

- *Existing open toolsets make building new open tools much, much easier.*

As a token of the many uses that a graphing tool like this may have, and also to illustrate the teachers' role in an open toolset development community, let me recount an experience. At a workshop to introduce some high school teachers to Boxer, I found myself demonstrating this little graphing tool to a mathematics and science teacher. His eyes lit up as he suggested that this would be a fine way to have his students get an intuitive idea of how a derivative relates to a function. He imagined that the student would draw a function, and the graphing tool would overlay its derivative. Figure 5 shows the essence of his wish, which we realized in about 5 to 10 minutes. It is a little program that scripts the graphing tool, named **g**, to (1) change the graph color to red, (2) draw a graph of the pair-wise differences in the data of the existing graph (which, presumably, the student had drawn), and (3) change the graph color back to blue for the next student-drawn graph. (**Differences** is a one-line program that a fluent Boxer or Logo programmer can create almost instantly.)



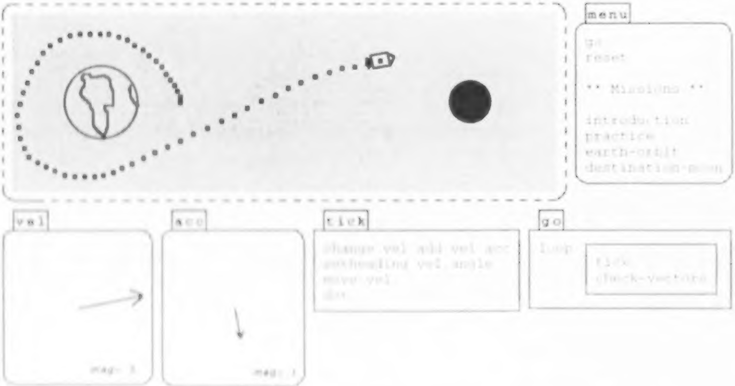
```
draw derivative
ask g
    change graph red
    plot data differences data
    change graph blue
```

Figure 5. A program that overlays the derivative on a drawn graph.

Here is the generalization: Good tools suggest interesting uses to teachers. Even if the teacher is not up to programming the modification or extension, another, more technically expert member of the community can often quickly extend an open tool in the way the teacher would like. Then it's off to the classroom to see how this works.

**Vectors**

This example is of a tool type that is both particularly powerful and also particularly characteristic of Boxer. Several years ago our group developed a course on physics for sixth grade students (diSessa, 1995). As part of the course development, a graduate student, Bruce Sherin, programmed an extension toolset that allowed us to use vector quantities in Boxer in pretty much the same way as one usually uses numbers. First, a special keystroke created a vector, which appeared as an arrow within a box. One can adjust any vector by dragging its tip with the mouse. Of course, vectors are scriptable, so any program can also command a vector to change. In addition, similar to the grapher, the flip side of a vector shows its coordinates, which are directly editable. Like any object in Boxer, a vector can be named. In addition, the toolkit provided commands to add vectors (**`add vector1 vector1`**), to multiply a vector by a number (**`mult vector number`**), and to have the vector have effects on other graphical objects (**`move vector`** causes a Boxer turtle to move the length of the vector in the direction of the vector).



```
tick
change vel add vel acc
setheading vel.angle
move vel
dot
```

```
go
loop
    tick
    check-vectors
```

Figure 6. A vector microworld in which students drag velocity or acceleration vectors to control a space ship.

Adding vectors to Boxer in this way is nearly the same as adding vector literacy to the range of competencies that can be fostered within this medium. It is about as powerful as having numbers on the keyboard and numerical calculations in a programming language.

Figure 6 shows a simple exercise microworld built using vectors. In it, students are requested to drive a space ship around the earth and moon by adjusting the acceleration or velocity of the ship in real time. The task is quite entertaining and challenging. It was a significant part of our instruction on how to understand such things as velocity and acceleration as vectors, but also on how to think about complex motions in terms of vectors.

One of the nice things about this microworld is that nearly the complete code for it is right there for students to inspect or copy. The procedure **`tick`** shows what happens "each tick of the clock." First, the velocity is incremented by the acceleration (i.e., the velocity is changed to its old value plus the acceleration). Then the space ship is directed to **`set`** its **`heading`** in the direction of the **`vel`**ocity's **`angle`** of pointing. The ship then **`move`**s according to its **`vel`**ocity, and, finally, the ship makes a **`dot`**. The other procedure, **`go`**, simply repeats **`tick`** over and over, along with **`check-vectors`**, which allows vectors to be changed while the space ship is also moving. **`Check-vectors`** is part of the vector toolset.

It should be evident how simple the vector toolset makes it for teachers or curriculum developers to make a very wide range of exercise microworlds for students. We used vectors dozens of times in the original physics course and in subsequent versions. For example, we made a simple tool that allowed students to analyze (in terms of velocity and acceleration) stroboscopic images they had created of tossed balls. In addition, most of what was given to students was exceedingly transparent. Students were expected to look at the code of the microworlds, like the space ship simulation in Figure 6, and learn from it.

Most impressive, vectors became thoroughly incorporated into the student culture. Many students made video games using vectors and the fragments of vector code they learned in exercise microworlds. Thus, a simple toolset that nonetheless introduces a powerful idea (vectors) flexibly into a computational environment like Boxer showed to us all the promise that we feel open toolsets may have in many other instances: evident utility to curriculum designers, teachers, and also students.

- *New graphical/computational objects, like vectors, may be among the most flexible and powerful of open tools for curriculum developers, teachers and students.*

In case you believe vectors are a special case—and certainly they are in some respects—Figure 7 shows a structurally similar tool. In this case, we are entering the area of genetics and evolution. The basic command, **`new-creature`**, creates an animal (they are called "scats") with a certain genetic makeup (Figure 7a). The animal's **`scale`** is a random number between 20 and 30 and its **`eye-color`** consists of one green allele and one blue one. If you flip the scat's box, shown in Figure 7b, you see that its insides consist of computational versions of its phenotype and genotype. In this case, the genotype **`scale`** translates simply into the phenotypical **`size`** of the animal. The scat's eye-color is a bit more complex. The two alleles, green and blue, interact, with the green allele being dominant. Thus a green-blue genotype results in a green phenotype. You can change the genotype directly "inside" the scat and see how the phenotype (automatically) changes.

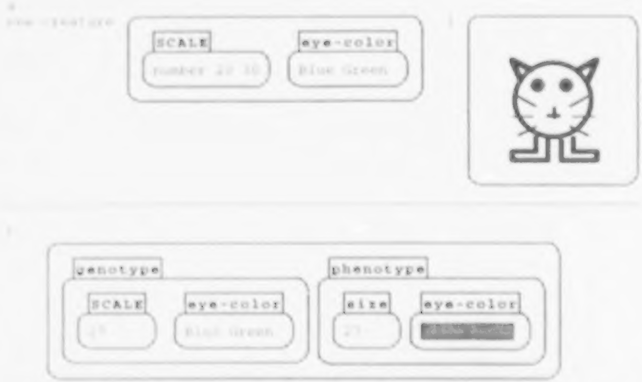

a
new-creature
SCALE
number 22 30
eye-color
Blue Green
b
genotype
SCALE
eye-color
Blue Green
phenotype
size
eye-color

Figure 7a: **`New-creature`** produces a graphical/computational object.
7b: The "flip side" of the object shows its computational structure.

With such a toolset, it is easy to set up a simple situation where scats breed with one another and produce offspring. Then you can "play Darwin" by selecting the scats you want to breed for the following generation. You can select for size, or eye color, or whatever you like. It happens that scats are easily extendible to add other genetic characteristics, and you can also change the little program that "expresses" (computes) the phenotype in terms of the genotype. Or write a different program for selecting scats to breed, and so on.

## Modeling Kits

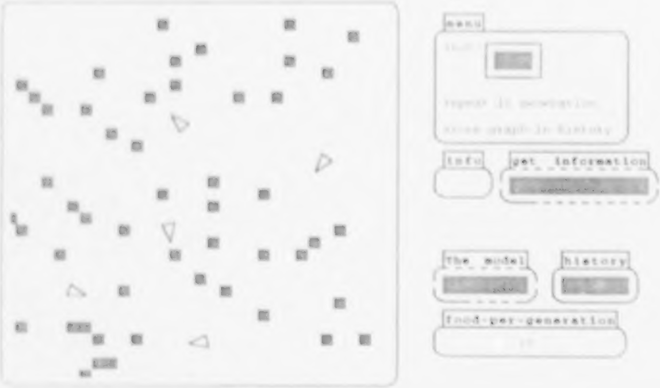
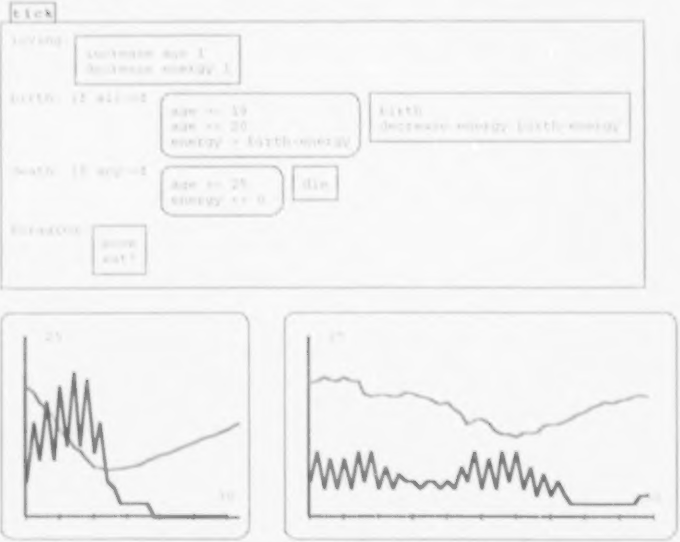

menu
info
get information
The model
history
food-per-generation

Figure 8: A configuration of a modeling kit that allows exploration of creatures (triangles) living in balance with their food (gray squares).

Figure 8 shows a toolkit to allow modeling of both ecological and evolutionary phenomena. On the left is a field in which creatures (the triangles—let's call them turtles) wander around, foraging for food (the gray squares—they are actually green). Each turtle has built-in properties, like its age and energy level (corresponding to stored calorie reserves). In addition, the kit has built-in functions to "**`birth`**" new turtles of any specification, to cause a turtle to **`die`**, to generate a certain number of food squares, and so on.

The configuration of the toolset shown is arranged to facilitate a certain kind of experiment. The **`init`** command sets up a certain amount of food and a certain number of turtles. In Figure 8, the gray Boxer boxes on the left are actually shrunken, but they may be clicked on to open and make available their insides.

The **`generation`** command runs the simulation for about one life-span of a turtle. **`Info`** provides helpful information to users while the models are running, similar to the greeting box in the calculator. **`Get-information`** contains resources to find out many things about the current state of the model: like the number of live turtles, their ages and energies, the amount of food available, and so on. **`The model`** contains the specification of how turtles work, and it is meant to be adjusted by students. The **`history`** box is simply an empty place in which students can keep notes about their various experiments. The last line in the menu just puts a graph of the population of turtles and amount of food into the history box. You shouldn't be surprised that the graphing tool described above is used to generate those graphs. So, again, available tools make building additional tools easier.



```
tick
living:      increase age 1
             decrease energy 1
birth: if all-of   age >= 19                birth
                   age <= 20                decrease energy birth-energy
                   energy > birth-energy
death: if any-of   age >= 25     die
                   energy <= 0
forage:   move
          eat?
```

Figure 9. The progam that defines behavior of the creatures, and two graphs of their population (black, jagged) and food supply (gray, smoother). The first graph shows "boom and bust," and the second a relatively stable, limited population.

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The top part of Figure 9 shows the initial turtle model—what the turtle does each "`tick`" of the clock. This has four parts: what happens during the normal course of living; the conditions for and affect of giving birth; the conditions for dying; and the foraging behavior (in this case, `move` is just some random motion).

The first graph in the bottom part of Figure 9 shows a typical behavior of this ecological system. It shows the turtle population (darker line) quickly but irregularly increasing while the food supply diminishes. This "boom" is followed by the expected "bust" part of the cycle; with food greatly diminished by many, hungry turtles, the large turtle population quickly dies off, leaving the food supply to replenish gradually in the absence of turtles. Keeping a stable population, in fact, is quite a difficult task, as I discovered myself when I first started to play with this modeling kit. My first solution was simply to put a cap on the population of turtles. I inserted a condition for birth that there be no more than 8 turtles. While this is artificial, it is effective, as shown in the second graph in Figure 9.

The first trial of this modeling kit with students actually was with student teachers in a secondary mathematics and science teacher credentialing program. The initial pedagogical difficulty was surprising. The student teachers of biology—who were quite steeped in the details of complex, real biological systems—rejected the possibility that a simplified mathematical model like this could tell anything about the real world. Even the "expected" boom and bust cycle was not convincing. I think this is not an accident of the population of teachers. Instead, knowing how mathematical modeling makes sense, in view of the simplifications it must make, is an important instructional goal.

The way this group of students completed their study illustrates some important points about open toolsets. In particular, they used the modeling kit in ways I had not anticipated. While I had provided means to get information about the live population, they wanted to do a post mortem on dead turtles to see why they died. Because the kit is open to inspection, they could easily delve into the internals of the kit to find and examine dead turtles.

The solution the students eventually found to solve the boom and bust problem is also illuminating. They noted that all the turtles age together, give birth together, and die of old age together. (From their post mortem, they discovered that almost all the turtles were dying of old age, not food insufficiency. The turtle population was dying, it turned out, not because lack of food was starving them to death, but because insufficient food kept them from having sufficient energy to give birth to a new generation!) So, the teachers changed the birth conditions to allow turtles to give birth over a longer age range. The resulting more diverse population, in fact, turned out to be significantly more stable! Before, the simulation produced a sequence of critical periods when the whole population is fertile together. At those times, the turtles must have sufficient energy to propagate, or the population perishes. Now, given a greater range of ages, "out-of-synch" turtles may weather tough times to propagate when food has regrown. Again, I did not anticipate this sort of change in the model. However, because the model was simply a program that the students could change, it could easily accept their innovative and excellent ideas.

### Reflections on the Research Program

I find the possibilities of open toolsets intriguing. (1) They appear to offer new directions for learning with excellent properties. (2) They may solve some difficulties in current computer-based instruction, like how to make software adaptable in the classroom. But open toolsets are also intriguing for pointing out how little we know about those possibilities. They even suggest that the generality of our current research paradigms is limited with respect to answering important questions. Let me explain.

I would divide a research program on open toolsets into three overlapping levels. The first is the one we know most about. It is the learning level. Most educational research on mathematics and science instruction documents and seeks to explain particular conceptual difficulties that students have in learning particular subject matter. This level finds a new context in the use of open toolsets. But, presumably, we can continue our research on student learning into these contexts. This includes both cognitive studies of individual difficulties and more socially oriented studies of learning-in-context.

The second level begins to outstrip current interests and paradigms of study. This is the level of "tools and activity structures." I mentioned earlier how important interest and personal (and communal) involvement is for the success of open toolsets. One example was the fact that vectors were taken up into the student culture, with consequent greatly extended learning time. Students simply cared to use vectors to accomplish goals they understood, so the properties of vectors in accomplishing those goals became important to them. A great deal of lip service is paid to motivation, especially in designing instruction. However, our understanding of the details of interest patterns, particularly how they can evolve, and their connection to competence and material support (like open toolsets) is extremely meager. I am struck at how little study is conducted that looks at any long term, felt-to-be-coherent engagement of students.

The third level focuses on "social niches and patterns of appropriation." Change in cultures and communities is a critical barrier that we face in improving education. There has been a significant amount of study of issues relating to this, largely under the rubric of reform. But almost all of this is in ancient correlational forms. There is hardly any structural study of the evolution of cultures and communities, and even less concern for the role of artifacts (like open toolsets) in facilitating change. Social theory and science, technology, and society studies focus on either stability, or on case studies of change. Neither are particularly apt to prepare us to design better for change. In particular, what current research can tell us whether open toolsets have better social properties than other forms of software? This is not necessarily an issue of turning our attention to societal, structural matters that bear on education. The process of appropriation, and *change* of community practices and values, happens in every classroom that changes to adapt to a different material support system.

### References

diSessa, A. A., Abelson, H., & Ploger, D. (1991). An Overview of Boxer. *Journal of Mathematical Behavior*, **10**(1), 3-15.

diSessa, A. A. (1995). The many faces of a computational medium. In A. diSessa, C. Hoyles, R. Noss, with L. Edwards (Eds.), *Computers and Exploratory Learning*. Berlin: Springer Verlag, 337-359.

Eisenberg, M. (1995). Creating Software Applications for Children: Some Thoughts About Design. In A. diSessa, C. Hoyles, R. Noss, with L. Edwards (Eds.), *Computers and Exploratory Learning*. Berlin: Springer Verlag, 175-196.

Epistemology and Learning Research Group of the MIT Media Laboratory (1991). *Constructionism Research Reports and Essays, 1985-1990*. Norwood, N.J.: Ablex Publishing Corporation.

Roschelle, J. & Kaput, J. (1996). Educational software architecture and systemic impact: The promise of component software. *Journal of Educational Computing Research*, **14**(3), 217-228.

# From Intuition to Inhibition - Mathematics, Education and other Endangered Species

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*An attempt is made in this paper to tie some specific aspects of mathematical behavior to some general aspects of human behavior.*

## 1. Do we really have goals in mathematics education?

In a letter to the committee of the international group of the psychology of mathematics education, Nicolas Balacheff (1996), a former president of PME, called PME members to question the aims and directions of their activities as PME members. *For human beings to be a twenty years old is still being a tender age, whereas for institutions it sounds as if it were the age of maturity,* he wrote. I absolutely agree. I just want to add that even if the average age of the founding group of PME were 35, then we are at least 55 now and this is certainly an age of maturity even for human beings. As a matter of fact, I have started questioning my activities as a researcher and a mathematics teacher long time ago. Balacheff's letter is only a good excuse to speak about it. What I am going to say is not a call for anybody to follow any direction presented in the paper. It is only a description of how I view mathematics education.

I started as a mathematician and mathematics teacher and then I switched to mathematics education. I realized that I need to know more about my students. Especially, how they think mathematically, how they acquire concepts, how they reason and how they solve mathematical problems. Thus, I became involved with cognitive aspects of learning mathematics. Questions of values were not asked at this stage. At early stages of our career, we are busy with survival. Only when the survival is taken care of, some of us start reflecting on the value aspects of their profession. This is in agreement with Maslow's hierarchy of psychological needs (Maslow, 1970, Chap. 4). Teaching mathematics is for some people a way to make a living. After having an economical security one gets to higher level of needs as self esteem and self actualization. These are closely related to contribution to society. Very often it is easier to believe that what you do is meaningful, if other people believe so. In order to make them believe so, you should make some contribution to society. Thus, the question why do we teach mathematics is raised. This is a special case of the question why do we teach anything? The answer is very simple and almost tautological: We teach in order to help people acquire knowledge. But why should anybody acquire knowledge? In the philosophy of education there are different ideologies concerning this issue. I will mention three of them as described in Lamm (1990).

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1. Social Instrumentalism. According to this ideology, knowledge is the most efficient tool of human beings in their struggle for survival. Hence, the knowledge which should be taught in schools is the knowledge which is most relevant and most important for survival. Herbert Spencer (1820 - 1903), who was a distinct proponent of this approach considered science as the core of the curriculum. If we adopt this ideology we can combine mathematics to science and thus justify the need to study mathematics. However, the claim that science and mathematics are crucial for the survival should be carefully examined. It is clear that they are crucial for the survival of the entire society but *this does not determine the percentage of people who should study science and mathematics and to what extent.*

2. Ritualism. According to this ideology, *there are categories of knowledge that posses intrinsic value and it is imperative to teach them. A person who does not have this knowledge is lacking in the basic essence of a human being.* We can tie this ideology with the movement of mathematics from a humanistic approach. The main difficulty of this ideology is the fact that there is no consensus about the knowledge supposed to be *the basic essence of a human being.* Thus, any curricular claim about mathematics as a humanistic subject is a minority claim. Minorities, in most societies, are *endangered species* which need legal protection and in most cases, legal protection is not enough to preserve the species.

3. Developmental Insrtumentalism. The focus of this approach is the individual. Knowledge is *an instrument that can help the person develop intellectual faculties.* According to this approach, *the desire to know is an inborn need; the expression of this need is curiosity. Learning out of curiosity is the way to acquire knowledge that fosters the learner's development.* The difficulty of this approach is the very assumption that *the desire to know is an inborn need.* Even if this assumption were true, it is clear that different people will have different desires to know and different curiosities. Therefore, why should everybody study mathematics? Developmental instrumentalism can be considered as complementary to ritualism. There are certain things which are essential for you as a human being. These things can satisfy your desire to know and thus lead you to self actualization. Maslow (1970, pp.48 - 51) considers curiosity, cognitive impulses, the needs to know and to understand, the desires to organize, to analyze, to look for relations and meanings as an *essential part of the human nature.*

Thus we have almost ready made answers to our question: Why do we teach mathematics? The fact that the educational community has ready made answers to some essential questions causes us quite often to act and to speak in *a rhetorical mode.* When you are in a rhetorical mode you say things which you are supposed to believe in. This may prevent you from critical thinking or even from noticing the facts. On the other hand, the ready made answer can be a claim in which you really believe.

If you look at what some people consider the greatest rhetorical document ever written in mathematics education (NCTM, 1989), you will find the following:

*Historically, societies have established schools to (1) transmit aspects of the culture to the young; (2) direct students toward, and provide them with, an opportunity* *for self-fulfillment* (p.2); *(3) In most democratic countries, common schools were created to provide most youth the training needed to become workers in fields, factories, and shops...Students also were expected to become literate enough to be informed voters* (p.3).

One can easily identify in these quotations the above mentioned three ideologies. To this the *Standards* add: *Today economic survival and growth are dependent on new factories established to produce complex products and services with very short market cycles* (p.3). This is a quite convincing economical description. What is *less* convincing is the following statement: *Although mathematics is not taught in schools solely so students can get jobs , we are convinced that in-school experiences reflect to some extent those of today's workplace* (p.4). It is not clear whether the document speaks about present schools or about future schools but it is quite clear (it seems to me) that the claim is more rhetorical then substantial. I myself have *never* seen a convincing evidence (neither experimental nor theoretical) that *in-school experiences reflect to some extent those of today's workplace.*

When I asked (Vinner, 1995) some 36 secondary mathematics teachers why do we teach mathematics, 26 responded that we do it in order to develop the students' thinking. When being asked what is achieved by teaching mathematics only 15 replied that the students' thinking is improved as a result of their learning mathematics. Is this the common gap between intentions and achievements? Is this the common gap between rhetoric and practice? If it is the first gap, then some mathematics teachers should be frustrated. If it is the second gap, then some mathematics teachers should be cynical. I assume that there are some mathematics teachers who are both frustrated and cynical.

When the achievements of the educational system are discussed, usually it is done within the framework of the goals. Hence, usually, some prosaic achievements are not mentioned at all. For instance, *a major achievement,* as keeping our kids busy with harmless activities during some crucial hours of the day, is generally not mentioned as a goal of the educational system. However, this is a major contribution without which society cannot economically function. Another *major achievement* of the educational system is that it supplies to many social systems a selection tool. Namely, high academic achievements increase the chances of a candidate to be accepted for higher studies or for a job, even if the knowledge acquired in the previous stage is not so relevant to the future stage. Mathematical achievements have a special role in the selection process. In many institutions the chances of a candidate to be accepted to study law, medicine, psychology and other disciplines which are not related or barely related to mathematics, increase a lot if the candidate scores highly in mathematics. As far as I know, there is no serious discussion of this problem in the literature of mathematics education . To support my claim I will quote one phrase from Jere Confrey (1995, p.3): ... *In the vast majority of countries around the world , mathematics acts as a draconian filter to the pursuit of further technical and quantitative studies...* So, why do we *really* teach mathematics? The question needs a serious discussion which should avoid dogmas, rhetoric or wishful thinking. Such a discussion has not taken place yet.

As I claimed earlier, the search for meaning of our activities occurs in most cases, if at all, after the lower needs are achieved (Maslow, 1970). On the other hand, there is a well known phenomenon that the more you use a word, the less you think about its meaning. Therefore, it might look a little bit strange that I am trying to examine the most common word combination in PME conferences which is: Mathematics Education. Especially, I would like to examine the word "education." This word is ambiguous. The Merriam Webster ninth edition claims that to educate is: ***1 a** : to provide schooling for* ***b** : to train by formal instruction and supervised practice esp. in a skill, trade or profession* ***2** to develop mentally, morally, or aesthetically esp. by instruction* ***3** to persuade or condition to feel, believe to act in a desired way or to accept something as desirable* (Webster's Ninth New Collegiate dictionary, 1986). In spite of this ambiguity, if you listen to people discourse or analyze various texts, especially when the word education is combined to science or mathematics, you cannot avoid the conclusion that the dominant meaning of the word is **1b** above. Education is almost synonymous with knowledge acquisition. An educated person is, in most cases, somebody who has at least a Bachelor degree in one of the common academic disciplines. Schools, the educational system and especially the mathematics education community, focus mainly on knowledge acquisition and not on values (mental, moral or aesthetically). As long as you teach mathematics in order to survive (namely, make a living) you have enough justification for doing it, but the moment you start thinking about the contribution you make to society, especially if you care about values, then some doubts may be raised. Even if we succeed to teach our students how to investigate a function or how to solve a system of linear equations in a final exam - is it really so important? Do we really contribute by that anything to the mental, moral or aesthetical development of our students, as the lexical definition suggests?

## 2. Concept Image, concept definition and inhibition

When I started analyzing my work in mathematics education research and the work of some of my colleagues I noticed that, in many of the cases, we compared students' concepts and thought processes with formal concepts and with mathematically valid thought processes. For instance, the students' conception of function was compared with the function concept held by the mathematical community (Vinner, 1983); the students' conception of some simple geometrical figures was compared with the one held by mathematics teachers and textbooks (Vinner and Hershkowitz, 1980), and the students' justification of a given mathematical claim was compared with the formal proof of this claim in mathematics (Vinner, 1983). In mathematics, since every notion (which is not a primary notion) has a definition, it is natural and easy to compare the students' conceptions with the formal mathematical definitions. This was the context where concept image and concept definition came in. They were used in order to perform the above mentioned analysis; namely, the comparison between the desirable concept and the concepts students have. On one hand, this comparison can be used in order to evaluate students' mathematical achievements. On the other hand, it tells us a lot about students thought processes and the factors which determine them.

The bottom line of the 'concept image - concept definition' studies was that in too many cases concept images control the students' thought processes and not the concept definitions, as desired. The concept image can be considered as part of intuition. A comprehensive theory of intuition in science and mathematics is given in Fischbein (1987). Here, I would only like to point at some essential elements of the intuitive mode of thinking which are relevant to my topics. Intuition is immediate, spontaneous, based on global impressions and does not use analytical processes. Concept images are evoked in our mind in an intuitive way. They are the immediate reaction of our mind to the concept name that we hear or see. By means of the concept images we are trying to work on given mathematical problem. It turns out that in some cases the intuitive mode of thinking just misleads us. In some contexts it is not desirable. The right move is to consult the concept definition, to ignore global impressions and, instead, to be analytical. That is also to *control* spontaneity and *avoid* the immediate reaction. Some people believe that acting spontaneously and intuitively is a virtue. I do not want to argue about this. I suggest that, at least in this context, a mode of action will be judged on the basis of its outcomes. If the outcomes are not desirable then the mode of thinking which produces them is not desirable. We should *inhibit* it. The word "inhibition" is heavily loaded, even more than "intuition." For many people it has a negative connotation. It also has a special role in some psychological theories. I would like to avoid all these and to use it, more or less, as it is used in everyday language. Therefore, let us look again at the dictionary. To inhibit, according to the Merriam - Webster, is: *to prohibit from doing something, to hold in check, to restrain, to discourage from free or spontaneous activity especially through the operation of inner psychological impediments or of social controls.* Here is the idea behind the title of this talk. In mathematical behavior, very often, we have to *inhibit our intuitions.* I am not claiming that we are supposed to suppress them for ever. Not at all. There is no harm in letting our intuitions start mathematical thought processes. In many cases it is even very fruitful. However, after the thought process has reached a certain point, we should control our intuitions, examine the outcomes, and use the analytical tools the discipline of mathematics provides us with.

## 3. The pseudo-conceptual behavior

All this has been said within what I call *the cognitive approach.* The assumption of the cognitive approach is that the student (or the person in consideration) is in a cognitive mode. Namely, he or she wants to know something about the world. Also, he or she believes that a certain claim about the world, the meaning of which they understand, is true. This approach is a product of the conception that the role of education is to provide knowledge to students. There is also the assumption that students collaborate with this idea in the way *we* understand it. Namely, they really want to acquire knowledge and they will do what they are asked to do in order to acquire knowledge. However, students' views of knowledge might be very different from us. We are adults who studied at least one classical discipline at a high level (college or university). In addition to that, we have reflected on what we did. We develop understanding of the structure of knowledge (at list in our specific discipline). We know what it means to think *conceptually* or *analytically* in mathematics. Students do not necessarily have all these. It is more reasonable to assume that many of them have other views. Views which are shaped by the practice they live in. This practice may shape the following view: Information is given to the students by teachers, books, TV, computers and other sources. There is also a system of questions and answers. Teachers pose questions to which students should give answers. This is done in class discussions, homework assignments and all kinds of written tests. The students get credit for their answers. It is very important to get credit. It is important because learning achievements are, among other things, *a draconian filter to the pursuit of further technical and quantitative studies* (Confrey, 1995). The educational system is, above all, a credit system. Hence, one of the students' main goal is to develop abilities to answer the typical questions posed to them by the educational system. The educational system seems to have a naive belief that the common way to answer the questions posed to students is by means of the knowledge these students have. But what if this is false? Many complaints against rote learning were raised in the past. The notion *learning without understanding* was introduced (see for instance, Rosnick and Clement, 1980). All these indicated that there are ways to get credit from the educational system for knowledge that you do not really have. Of course, in order to get the credit you must have a certain knowledge, but this is not the *desirable* knowledge (desirable - by the educational system). Let me call this knowledge - pseudo-knowledge. The word "pseudo" in this context indicates that there is something which is true knowledge and something else which is not true knowledge, by means of which one can get credit, usually achieved by means of true knowledge. By the way, the adjective "pseudo" can be attached to many "entities", denote them by X, in case it is possible to explain what true X is. Just try to substitute X by art, friendship, culture, meaning, drama, wisdom, etc.. But if you wish to speak about pseudo-X, you may face a serious problem. You may discover that there is no consensus about, what *you* wish to call, true X. In this case, there will also be no consensus about pseudo-X. Relativism, constructivism and post modernism are potential enemies of the attempt to speak about pseudo-entities. However, I can defend it at least within the framework of mathematics and science education. I believe that in the community of mathematicians and scientists there is a consensus about what can be considered true mathematical knowledge or true scientific knowledge.

Since the educational system is also a credit system, it is important for it to identify pseudo-knowledge. Its intention is to give credit for true knowledge and not for pseudo-knowledge.

As to those who use pseudo-knowledge in order to get credit - there are two possibilities: 1. They know that they use pseudo-knowledge in order to get credit. 2. They do not know it.

From the educator's point of view, the first situation is a little easier than the second one. In the first situation mainly cheating is involved. It is not this kind of cheating which implies punishment in our society. Basically, it is pretending. You pretend to know when you do not know, and you know that you do not know. You do it because it enables you to get some credit. Of course, the assumption is that it is easier to acquire pseudo-knowledge than true knowledge. Thus, in a way, there is a value problem here. You can even consider it as a moral dilemma: *To pretend* and to get some credit *or not to pretend* and get zero credit? The vast majority of people that I know, including myself, will solve this dilemma without much hesitation: We will pretend. The price for not pretending is too heavy, especially if your future depends on the credit you are so desperate to get; future in terms of further studies or a job. Pretending in these situations is considered as following *the rules of the game*. However, the mere fact that we notice a moral dilemma has some moral value.

For most of us ,dealing with moral values in our classes is an undesirable and even awkward situation. This is, perhaps, because it reminds us preaching. *Papa, don't preach,* sings Madonna. May I use this as an indication that preaching is not popular? There is also the common view that if you preach, then you must be perfect and, of course, nobody is perfect. Perhaps, a more appropriate starting point to a moral discussion is the above general claim that nobody is perfect. *I am not an angel,* claimed James Hewit, the lover of Lady Diane in his famous TV interview. By admitting that, he located himself immediately inside the domain of moral values. Admitting that, also made it possible for him to discuss moral values. If you prefer a reference with a more institutional credit, just recall *I Kings 8,46* which claims: *There is no man that sinneth not.* My point is that we cannot avoid dealing with values when we teach. The question how to do it should get much more attention from educators and educational researchers.

I would like to return now to the above situation in which an individual is not aware of the fact that he or she uses pseudo-knowledge in order to get credit. As I said earlier, this might be a harder situation than the first one. It is harder from the cognitive point of view, because the individual has no idea what a true knowledge is. He does not pretend. He assumes. Namely, he does not give *a false appearance*. He assumes he gives *a true appearance*. If this happens in our classes it can be considered as failure in teaching the subject matter; presumably, our main goal in education. Again, I am not going to suggest a cure to this disease. I am still at the diagnosis stage and I would like to make now a general comment about credit: We are all credit edicts, credit - in the most general sense of the word. *All people in our society have a need for high evaluation of themselves and for the esteem of others* (Maslow, 1970, p.45, with some omissions). Other words mentioned by Maslow at the same context are: *reputation, prestige, status, dominance, recognition, attention, importance and appreciation.* I will refer to all these by the single word "credit." So, it is not only within school that we look for credit. It is everywhere: in the family, with friends, at a cocktail party, at work and, of course, in the field of academic achievements. Being so 'desperate' about credit, no wonder we get it sometimes, intentionally or unintentionally, on the grounds of false assumptions.

All this was generally speaking. Let us look now at mathematics education.

There are two common activities in which mathematics students and teachers are very often engaged: 1. Students discuss with their teachers (or in writing) the meaning of certain mathematical notions. 2. Students solve routine mathematical problems.

In the first activity, concepts are involved. Students are expected to think about concepts, their meaning and their interrelations. If they really do it, they are in a conceptual mode of thinking. If they do not, but in spite of that succeed to produce answers which seem to be conceptual, then I will say that they are in a *pseudo-conceptual* mode of thinking.

In the second activity, if the students act the way they are expected, certain thought processes, which deserve the title *analytical,* should occur. If they do not, but the students succeed to make the impression that they are analytically involved in problem solving, then I say that they are in a pseudo-analytical mode of thinking.

Thus I suggest two notions (pseudo-conceptual and pseudo-analytical) which are strongly related, but also different, because they are applied to two basically different types of mathematical activities. Take, for instance, the following excerpt from a 12th grade mathematics class:

Teacher: *Give a recursion rule for the sequence:* $a_n=n^2$ Students: ... Teacher: *What is a recursion rule?* Student A: $a_2 - a_1$. Student B: *2n-1*
Student C: $a_1 + a_1 = a_2$ Teacher: *How should you express* $a_{n+1}$? Student D: *1.*
Student E: *n*

The excerpt starts as a routine problem solving situation. Very soon, it turns into a discussion about concepts. It happens because the students are stuck. The teacher believes that if the students understand the meaning of the notion *recursion rule* then they will be able to perform the task. The answer which the teacher expected was: $a_{n+1}$ *should be expressed by means of* $a_n$ *and n.* Thus, superficially, the students were quite 'close' to this answer. However, there was very little thinking involved in the above discussion. It was mainly blind guessing. On the other hand, if you do not know the particular topic (recursion, in this case ) you might get the impression of a meaningful discussion. The characteristic of the students' reactions is the following: The terms expressed by the teacher evoke in their mind certain associations. Since they lack understanding of the topic, they cannot examine these associations and tell whether they constitute a correct answer or not. Thus, there are two alternatives. The first one is to remain silent. The second one is to express what they have on their mind. At least some students do not think they risk anything by telling the teacher what they have on their mind. If you usually do not practice critical thinking or you lack reflective abilities, you cannot consider the uncontrolled reaction as negative. If you lack the mechanism of examining your associations and determining whether they make any sense in a given situation, then remaining silent is not an option. In the above behavior the students miss only one stage: the control stage. All of us have associations when we hear or see a certain notion. We cannot control our associations. They are the internal reaction to a given stimulus. However, we, sometimes, can control our behavior, since it is an external reaction to the stimulus. Here comes again the idea of inhibition. To *control* is closely related to *inhibit.* A dominant feature of the pseudo-conceptual thought processes is the uncontrolled associations which fail to become a meaningful framework for further thought processes. I would say that in true-conceptual thought processes, certain ideas are combined with other ideas, whereas in pseudo-conceptual thought processes words are combined with words. Most of us have some experience with situations in which we combine certain words or symbols and express them without knowing exactly what they mean if they mean anything. These might be test situations, professional situations and certainly a cocktail party situations. Take, for instance, the following fictitious (but most typical) cocktail party conversation:

Person A: *Katz is a post modernist.* Person B: *I hate post modernism.*
Person C: *It's not my taste either but you can't ignore the impact it has on our culture.*

It is quite possible that the people involved in the above 'most interesting' conversation know more or less what post modernism is. But it might well be that one of them or even all of them have no idea what it is, and still talk about it in such a way that it sounds like a meaningful conversation. A very simple analysis can show how it happens. In case the people do not know the meaning of post modernism, they can still identify the contexts in which the term is used and they can use some *keywords* and some *stereotypical phrases* related to the term. With these, such a conversation can take place. These are the tools of a potential deceit.

Here is another mathematics education example (a 10th grade class).

Teacher: *What is the distance between two points?* Student A: *The slope.*
Student B: *A straight line.* Student C: *A segment.* Teacher: *... The distance between two points is the length of the segment connecting the two points.*
Student C: *But this is exactly what I have said.*

The reaction of student C is quite typical for the pseudo-conceptual mode of thinking. There are no clear distinctions between slightly different elements. In this case, the slightly different expressions define essentially different mathematical entities, a geometrical entity (the segment) and a number (the length of the segment). And again, as in the previous examples, the use of key words related to the situation is quite striking.

One of the literary giants who considered pretense as a central theme of human behavior was Moliere. Just recall Le Bourgeois Gentilhomme, Le Malade Imaginaire and Don Juan. The first two tried to claim credit which they believed they really deserved. Don Juan, on the other hand, tried to get credit which he knew he did not deserve. All the three are relevant the point I am trying to make about mathematics education. Let us consider first the most famous excerpt from Le Bourgois Gentilhomme.

Mr. Jourdain: *...I am in love with a lady of quality and I want you to help me to write her a little note I can let fall at her feet.* Philosopher: *Very well.* Mr. Jourdain: *That's the correct thing to do, isn't it ?* Philosopher: *Certainly. You want it in verse no doubt ?* Mr. jourdain: *No. No. None of your verse for me.* Philosopher: *You want it in prose then?* Mr. Jourdain: *No. I don't want it in either.* Philosopher: *But it must be one or the other.* Mr. Jourdain: *Why?*

Philosopher: *Because, my dear sir, if you want to express yourself at all, there's only verse or prose for it.* Mr. Jourdain: *Only prose or verse for it?* Philosopher: *That's all, sir. Whatever isn't prose is verse and anything that isn't verse is prose.* Mr. Jourdain: *And talking, as I am now, which is that?* Philosopher: *That is prose.* Mr. Jourdain: *You mean to say that when I say "Nicole, fetch me my slippers" or "give me my night -cap" that's prose?* Philosopher: *Certainly, sir.* Mr. Jourdain: *Well, my goodness! Here I have been talking prose for forty years and I have never known it.*

Note that the entire dialogue starts with a value problem. Mr. Jourdain, a married gentleman, asks his teacher to help him writing a love letter to a woman who is not his wife. *That's the correct thing to do, isn't it ?* Asks Mr. Jourdain. And the teacher's answer is: *Certainly.* (As in common mathematics classes, values are not discussed. Teaching mathematics is our only business, isn't it?) The above conversation, till a certain point, sounds like meaningful communication. (*You want it in verse no doubt ? - No. No. None of your verse for me.*) Only at a crucial point it turns out that the student has no idea of the topic discussed. (*No. I don't want it in either.*) This is quite similar to the above excerpts where mathematical topics were discussed by real teachers and real students. Mr. Jourdain is trying to pretend. He cannot be successful for ever. On the other hand, is it so important to know what prose means? Forty years he never heard about prose. In spite of that, he was quite successful in business. Isn't this more important than formal education? Mr. Jourdain would try later on to impress his wife by the knowledge he had acquired. He *really* believes that the use of some key words makes him an educated person. In the context of mathematics education, notions like function, derivative or infinity will be at stake instead of prose and poetry. The question is: are they really so important to the majority of our students?

Contrary to Mr. Jourdain, Don Juan is *simply lying.* But the method is essentially the same: the use of key words and stereotypical phrases which are relevant to the context. And what are the key words and the stereotypical phrases which are relevant to the context of Don Juan? A lie has a better effect if it comes with Mozart's music. Therefore, let us look at Da Ponte's *Don Giovanni* instead of Moliere's *Don Juan.* Don Giovanni is trying to seduce a young lady, Zerlina, who is engaged to a young man, Masetto. It all happens on a street in Seville. After promising marriage to Zerlina, Don Giovanni points with his finger at a beautiful house on the street, claiming that it belongs to him. The following dialogue takes place:

Don Giovanni: *There we'll hold hands/ And you'll say "yes."* Zerlina: *I would like to and I wouldn't./ My heart trembles a little./ I would be happy, but/ He can still deceive me.* Don Giovanni: *Come my delight.* Zerlina: *I am sorry for Masetto.* Don Giovanni: *I'll change your life.* Zerlina: *My strength is falling.* Well, three replicas and her strength is already falling.

Thus, it seems that with some talent - it is not so hard to pretend, and without minimal caution - it is quite easy to be deceived. But don't we wish to be deceived, especially when it comes to our students' achievements?

## 4. The pseudo-analytical behavior

Let us move now to the second kind of activities which are so common to mathematics classes, namely, solving routine problems. Richard Skemp (1976) pointed at the difference between *knowing why* and *knowing how*. He recommended that our goal in mathematics education will be *knowing how*. Let us ignore wishful thinking for a moment and look at the practice. The most critical situations in mathematics learning (mainly, tests and test preparations) are routine problem solving situations. In these situations problems are posed to the students, and they are supposed to choose the solution procedure suitable for each given problem. The focus is not on *why* a certain procedure gives you the desirable result. The focus is on *which* procedure should be chosen in order to solve the problem, and then, *how* to carry out that procedure. The intellectual challenge is the correct selection of the solution procedure. The student is expected to be *analytical*. *Analytical* is not such a simple notion. A lot can be said about it. Because of space restrictions I will say only the following: To be analytical when solving a routine problem means to identify correctly the type and structure of the given problem, to select a procedure which is suitable for solving the problem and to apply this procedure to the given problem in order to get a solution. For instance, consider the problem: *Find the area of a rectangle the sides of which are 7cm and 5cm*. An analytical thought process for its solution may be the following: *This is an area problem. The area in consideration is that of a rectangle. In the given problem the two sides of the rectangle are given. The area of a rectangle is the product of it sides. The sides in this case are 7 and 5 and, therefore, the area is 7*5, namely,* $35cm^2$.

The above paragraph does not necessarily describe what really happens in the mind of somebody who solves the above problem analytically. But we can consider it as a schematic model for the process. The above problem can also be solved by the following thought process: 1. *This question is similar to questions we solved when we studied about the area of a rectangle. These questions were solved by multiplying the two numbers given in the questions. The two numbers in this case are 7 and 5 and therefore the result is 7*5, namely 35.*

Contrary to the analytical thought process, the elements of this process emerge from unspecified memories, rough identifications and vague procedures (instead of *the area of a rectangle is the product of it sides* there is the phrase: *the solution is obtained by multiplying the two numbers*). This mode of thinking I would like to call *pseudo-analytical*. A pseudo-analytical thought process may produce a correct answer as in the above example. On the other hand, assume that the following question is posed to a student who is in a pseudo-analytical mode of thinking:

*Find the area of a rectangle one side of which is 7 cm and the perimeter of which is 24 cm.* Then a possible line of thought for solving this problem is: *The problem looks like an area of a rectangle problem. In such problems the answer was obtained by multiplying the two numbers in the problems. The two numbers in this case are 24 and 7. Therefore the answer is 24*7, namely, 168.*

My claim is that there are many pseudo-analytical thought processes used by many students for many routine problems which constitute the majority of the common mathematical exams. It is important to investigate these processes because of two main reasons. First, it is basic research. Second, and more important, if we want to encourage our students to be analytical, we should explain to them the difference between *analytical* and *pseudo-analytical*. Therefore, we should know their pseudo-analytical thought processes. The moment one knows the difference between analytical and pseudo-analytical he or she can reflect about their thought processes, abandon the *pseudo-analytical* and follow the *true-analytical*. I say abandon *the pseudo-analytical* because usually the pseudo-analytical comes first. It is an immediate reaction to some key words in the question or to its superficial structure. Returning to the title of the paper, we are supposed to inhibit the pseudo-analytical thought processes at a certain point and consult the analytical ones. When I say "inhibit" I do not mean that we are not supposed to let them occur. As I said earlier, we cannot control our associations; they are our immediate and intuitive reactions of the mind. We can, *perhaps*, control our response to a given stimulus, namely, our behavior. Thus, again, we are moving from intuition to inhibition.

Here is an anecdotal example: My friend, a botanist asked her boss for a salary raise. The boss told her he could not raise her salary but he could offer her working *four* days a week instead of *five*. In away, this is a salary raise. She wanted to know what the percentage of this raise was. So, she came to me to get the answer. As somebody who teaches mathematics I thought my answer should be given in no time. *It is 20% raise*, I said and then, not being sure about the correctness of the answer I added: *Just a minute, let me reconsider this...* After awhile I gave her the correct answer. When analyzing this event I discovered all the typical elements of a pseudo-analytical behavior. I put myself under pressure - to give an answer in no time. I wanted to impress her (that is the credit story). I used the two numbers mentioned in the question in a similar way to the student calculating pseudo-analytically the area of a rectangle. Something like: *The numbers are 4 and 5. 4 is 80% of 5. From 80% to 100% - that's 20%.*

An interesting question is to what extent the pseudo-analytical behavior is common among our students. As I explained earlier, the pseudo-analytical thought processes lead us very often to correct answers and therefore, it is hard to detect them. However,, the following example will add a statistical aspect to this paper. The following two questions were given in the Third International Mathematics and Science Study(TIMMS, 1996):

*1. Which of the following numbers is the smallest?*

*a) 1/6   b) 2/3   c) 1/3   d)1/2*

*2. Which number is the greatest?*

*a) 4/5   b) 3/4   c)   5/8   d)   7/10*

There are several algorithms for comparing fractions. The most general one is the common denominator algorithm. The other algorithms are suitable for specific cases. For instance: (I) if the numerators are equal then the smallest fraction is the one with the greatest denominator. (II) If the numerators are equal then the greatest fraction is the one with the smallest denominator (these are not part of the general algorithm for comparing fractions). An analytical thought process will examine the situation in the task, will examine the pool of algorithms available for this type of task and will select a suitable algorithm for the specific given task. A pseudo-analytical thought process might start with a vague memory of algorithms. Perhaps: (I') The smallest fraction is the one with the greatest denominator. (II') The greatest fraction is the one with the smallest denominator. If algorithm (I') is applied in order to deal with the first question then the correct answer is obtained. Therefore, we cannot tell the percentage of students who use pseudo-analytical thought processes when dealing with this question. It turned out that 87.7% of a representative sample of Israeli 8th grade students (N=1415) answered the first question correctly. On the other hand, if a student uses algorithm (II') in order to deal with the second question, he or she will choose distracter (b), a wrong answer. In fact, 38.8% out of the above sample chose distracter (b). Only 45.6% answered the question correctly. If you add 45.6 and 38.8 you get 84.3 which is quite close to 87.7. Does this mean that about 39% who got the right answer in the first question got it by using pseudo-analytical thought processes? It is only a speculation, because the students were not asked to explain how they reached their answers. In another study, with much smaller sample of 8th graders in Israel, students were also asked to explain how they reached their answer. Many of them mentioned (II') as their method.

As I claimed earlier, the pseudo-conceptual and the pseudo-analytical modes of thinking occur in everybody. Sometimes, we are unaware of them (very often in these cases, we do not know what true conceptual or true analytical means); sometimes we are aware of them, and in these cases we are probably trying to achieve some goals which probably cannot be achieved otherwise. One of the greatest mathematicians of all generations is Leonard Euler (1707 - 1783). Surprisingly, he was using pseudo-conceptual argumentation, consciously and unconsciously. According to some historians of mathematics (Bell 1953, pp.159 - 160), when Euler was invited to a public debate with the French atheist Diderot (1713-1784) at the court of Catherine *the great* (1729 - 1796), he said: $\frac{a+b^n}{n}=x$, *hence, God exists.* To somebody who does not know mathematics, it might look like a real mathematical argument and of course, nobody can argue about the validity of a statement which has been mathematically proved. Poor Diderot asked Catherine's permission to quit. Speaking about moral aspects of mathematical behavior, this story is not recommended as a role model. This is an inhibition of arrogance failure. Here, Euler used deliberately a pseudo-conceptual argument to achieve superiority in a theological debate. On the other hand, he was involved several times, probably without knowing it, in pseudo-conceptual mode of thinking when he made his great mathematical discoveries about infinite series. This is not the place to establish my claim with examples. In order to support it, I will just quote Bell's comment about this topic (p.165): *The curious thing is that, although Euler recognized the necessity of caution in dealing with infinite processes, he failed to observe it in much of his own work. His faith in analysis was so great that he would sometimes seek a preposterous 'explanation' to make a patent absurdity respectable.* Returning from Euler to our own mathematics students, don't we see them sometimes seeking *a preposterous 'explanation' to make a patent absurdity respectable?* Thus pseudo-conceptual thought processes are important and even necessary for geniuses, not only for ordinary people. Therefore, it will be a mistake to try to eliminate the pseudo-conceptual mode. What we should do is to encourage the true conceptual mode to follow it.

One might argue that it is wrong to criticize Euler for being pseudo-conceptual because the true conceptual mode had not been established until Gauss (1777 - 1855) and Cauchy (1789 - 1857). Such an argument relates to the point I made earlier which was: How can we speak about pseudo-X if there is no definition of true X. It turns out that in many cases certain people agree about different issues although there are no definitions at hand. This might be a result of taste and style. The typical features of the pseudo-conceptual style is its *vagueness* and *obscurity*. Again, I have to admit that these are relative notions. Recently, a funny scandal, named Sokal's hoax, shook the academic world. Alan Sokal, an NYU mathematical physicist, submitted a sham paper to a journal of cultural studies by the name of *Social Text*. The paper underwent a reviewing procedure, was accepted for publication and was eventually published (Sokal, 1996a). After it was published, Sokal published another paper (1996b) in which he revealed the hoax. This was quite embarrassing for the editors of Social Text. But beyond the embarrassment, the way I see it, there is the contrast between the true conceptual and the pseudo-conceptual, even if there is no definition for true conceptual in the field of cultural studies. Some people will be enchanted by the title: *Transgressing the Boundaries - Toward a Transformative Hermeneutics of Quantum Gravity.* On the other hand, some people, when facing death, will draw *some consolation that they would never again have to look up the word 'hermeneutics' in the dictionary* (Weinberg, 1996, p.11). It seems that we deal with two human species here. C.P. Snow spoke about two cultures (Snow, 1959). Is mathematical thinking the element which makes the difference between these two species? Probably not. We have the famous example of Wittgenstein and also some other, less famous, examples. Being a mathematician does not necessarily imply the use of mathematical style of thought in other domains. Nevertheless, this is the essence of teaching mathematics beyond the technical content, the way I look at it.

## 5. Summary

In this paper I tried to draw some similarities between expectations from desirable mathematical behavior and desirable moral behavior. The similar feature is, what I call, from intuition to inhibition. Intuition in mathematical thinking is the immediate global reaction to a mathematical stimulus. Intuition in moral contexts is the impulse or the first tendency to act in a certain way. If we inhibit these for a minute, reflect on them, replace them if necessary according to some moral principles and then act - the result is a desirable moral behavior. Don Giovanni is a good negative role model. Graham Vick, the director of the Israeli production of Mozart *Don Giovanni* in Tel Aviv, 1994, said in a public interview that for him, Don Giovanni is an example of thoughtless behavior. Being an English man he was certainly aware of the ambiguity of "thoughtless": 1. careless. 2. lacking concern for others (Webster, as above). However, Vick did not elaborate on this point. If you focus more on Mozart's *Don Giovanni* you discover that the first name of the opera was *Il Dissoluto Punto* which is: The Libertine Punished. "Libertine", as many other English words, is ambiguous. One of its meanings is: a person who is unrestrained by convention or morality (Webster). Hence, the message is to restrain or, in another word, to inhibit.

Recently, when research grants for mathematicians underwent severe cuts, some mathematicians claimed they started to feel like dinosaurs (*Pure mathematicians are viewed in many quarters as dinosaurs, and the world will watch us sink into the tar pits with hardly a passing tear.* Krantz (1995)). On the other hand, it seems that mathematics educators still feel safe. I believe that there is no ground for this complacency. Because of several reasons, which I mentioned in the beginning of the paper, the present worldwide educational system still gives a lot of emphasis on mathematics. If we do not stand and deliver, this won't last for ever.

It would be better if we behave like an endangered species and not like arrogant carnivores. There are some other endangered species like classical music, poetry, art, etc.. Perhaps, in order to survive, one has to pretend from time to time. But can we pretend forever? After 20 years of PME, Balacheff (1996) asks: *we may have to question what allows us to pretend ...?* If you pretend too long you might discover that you cannot get rid of your mask. This happened to Jack Lemon in Billy Wilder's *Some Like it Hot*. In order to survive he runs away dressed like a woman. He or she (here it is the same person) meets an old millionaire who falls in love with him or her. They plan to escape together in a boat. Finally, the time has come to tell the millionaire the unpleasant truth. Lemon invents all kinds of excuses why she (or he) cannot get married to the millionaire. All of them are rejected. Therefore, the real reason should be discovered. *I am a man*, says Lemon hysterically. And the answer of the millionaire is: *Nobody is perfect.*

I also claimed it in the beginning of my paper. Nobody is perfect, neither in mathematical behavior nor in moral behavior. However, this statement does not have to serve as a justification for the current state of affairs. It might well be a good starting point for a change.

## References

Ballacheff, N. (1996). *A letter to the president of PME.*

Bell, E. t. (1953). *Men of Mathematics*, Penguin Books, Vol. 1.

Confrey, J. (1995). Student Voice in Examining "Splitting" as an Approach to Ratio, Proportions and Fractions, *Proceedings of PME19*, Recife, Vol.1, 3-29.

Fischbein, E. (1987). *Intuition in Science and Mathematics - An Educational Approach.* Reidel Publishing Company.

Krantz, S. (1995). Math for Sale, *Notices of the AMS,* 42 (10), 1116.

Lamm, Z. (1985-1990). Knowledge: Conceptions of and Impact on the Curriculum, *The International Encyclopedia of Education,* Pergamon, 3148-3151.

Maslow, A. H. (1970). *Motivation and Personality,* Harper &Row(second edition).

NCTM (1989). *Curriculum and Evaluation Standards for School Mathematics,* Reston, Virginia, USA.

Rosnick, P. and Clement, J. (1980). Learning without Understanding: The Effect of Tutoring Strategies on Algebra Misconceptions. *Journal of Mathematical Behavior* **3**, 3 - 27.

Snow, C. P. (1959). *The Two Cultures and the Scientific Revolution.* Cambridge University Press.

Skemp, R. (1976). Relational Understanding and Instrumental Understanding, *Mathematics Teacher* **7** , 20-26.

Sokal, A.D. (1996a). Transgressing the Boundaries - Toward a Transformative Hermeneutics of Quantum Gravity, *Social Text* (Spring/Summer), 217-252.

Sokal, A. D. (1996b). A Physicist Experiments with Cultural Studies, *Lingua Franca,* 62-64.

TIMSS (1996). *Mathematics Achievements in the Middle School Years (IEA's Third International Mathematics and Science Study).* Boston College, MA, USA.

Vinner, S. (1983). Concept image, Concept Definition and the Notion of Function, *The International Journal of Mathematical Education in Science and Technology* **14**(3), 293-305.

Vinner, S. (1983). The Notion of Proof - some aspects of students' views at the senior high level, *Proceedings of PME7,* Jerusalem, 289-294.

Vinner, S. (1995). Teaching Mathematics as an Educational Task - teachers' views about some aspects of their professional lives, *Proceedings of PME19,* Recife, Vol. 3, 328 - 335.

Vinner, s. and Hershkowitz, R. (1980). Concept Images and Common Cognitive Paths in the Development of some Simple Geometrical Concepts, *Proceedings of PME4,* Berkeley, 177-184.

*Webster's Ninth New Collegiate Dictionary* (1986). Merriam - Webster Inc. Publishers, Ma, USA.

Weinberg, S. (1996). Sokal's Hoax, *The New York Review,* Aug. 8, 11-15.

# PLENARY PANEL

**Plenary Panel**

## COGNITION, TECHNOLOGY, AND CHANGE

| | |
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| *Coordinator* | Kathryn Crawford |
| *Panelists* | Janet Ainley<br>Nicolas Balacheff<br>James J. Kaput |

# DISTRIBUTED COGNITION, TECHNOLOGY AND CHANGE:

## Themes for the Plenary Panel

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### Introduction

This paper takes a systemic approach to the challenges presented by the emergence of new technological artefacts in human activity which change the ways in which mathematics is made in the community and the ways in which mathematics is used. These changes are evident in every field of activity - in sport, in the creative arts and in commerce.

In contrast to the extensive technological change, schools and educational practice are stable institutional structures. The nature of 'school mathematics', the order of the curriculum and the social order of schooling has not fundamentally changed in most countries of the last twenty years. In particular, the cognitive demands of learning mathematics at school and undergraduate level remain firmly focussed on capabilities to demonstrate operational knowledge of approved procedures and axioms. Despite efforts in educational reform, assessment of mathematical understanding remains an elusive goal for many teachers. In the community the set procedures and routines are the work of machines. Employment opportunities for routine, repetitive tasks are reduced in the wake of automation. On the other hand, ever more complex transactions and other forms of social organisation have become possible though the support of information technology. More complex mathematical reasoning and understanding are needed by people in assessing their justice or desirability. The tension between educational practice and the needs of a community which uses information technologies has become particularly acute since the recent rapid development and wider accessibility of the internet and networked interactive multimedia. These changes suggest a new focus for research in mathematics education:

> **How can the tensions between historically based and stable forms of educational practice and the challenges and opportunities of new forms of mathematical activity and development presented by new technologies be resolved?**

The tensions between changing community activity and stable educational practices place pressures on students, teachers and

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educational researchers. The changes raise serious questions about the nature of education, the kinds of mathematical knowledge that are needed by learners of all ages, the kinds of educational settings that can and should be made available for the twentyfirst century, and the kinds of learning experiences and mathematical knowledge that are needed by humans in an era when machines commonly carry out the set procedures that have dominated school mathematics in the past.

The implied shift suggests a shift in the nature and distribution of cognitive activity in learning communities in mathematics education. This paper explores the issues surrounding such a shift. Such an exploration involves investigating a wider set of interacting issues than the small set of interactions, involving students, their teachers and their artefacts, that have been the focus of much research on learning mathematics in technologically rich environments.

**Hierarchies and Networks**:

A systemic approach to learning activity including the needs and goals of all the participants and the ways in which cognition and learning are affected by notions of "appropriate" behaviour in particular social situations. In hierarchies the virtual social organisation as it is understood has an existential meaning that is focussed on impersonal technical implementation (Crawford, 1986; 1995; 1996, Crawford & Deer, 1993, Crawford & Adler 1996). For example, many teachers regard it as their responsibility to set *suitable* mathematics problems, *show* students how to solve them, *tell* them about socially approved forms of mathematics that are set for the teacher in curriculum documents, provide opportunities for students to *practice exercises*, *evaluate solutions*, and *assess* that students are able to demonstrate that they have internalised and can reproduce a set of mathematical axioms and procedures. In general curriculum experts have already prescribed the problems or approved a text book, the methods of solution, the method of assessment and evaluation. For the students the strategic focus of their attention is on fast, accurate, automated implementation of procedures and memorisation of axioms and information. For *successful* students entering university mathematics courses more that 80% view mathematics in fragmented ways and mathematics learning as involving practicing procedures until they can be performed fast and accurately and memorising axioms and other information and definitions (Crawford, Gordon, Nicholas & Prosser, 1996). For teachers the strategic focus of their attention is on their responsibilities as implementers of a set curriculum. With a largely unelaborated concept of learning it is hardly surprising that in a recent survey (Crawford in Press) more than 82% of teachers at primary and secondary level conceived of information technology generally and computers in particular as just another thing to be taught. Less than 10% of the sample expressed the possibility that new technologies might afford new forms of learning and none thought that such forms of learning were practical in a school setting.

Importantly, both groups are largely unaware of the influences of the setting of their joint activity. Both students and teachers believe that they work as individuals.

This *virtual* social setting, which has cultural historical (Vygotsky 1978) origins, is widely understood by parents, students and teachers as *education* and has been a powerful force in lifting levels of numeracy in ways that have supported community mathematical needs in the past when the majority people carried out repetitious arithmetic tasks in commerce and clerical positions. However, the continuing focus on operational planes of activity is as odds with the realities of increasingly technically rich vocational and domestic settings where machines carry out these procedures and people increasingly need to be able to define problems mathematically and assess the implications of the solutions produced.

In developed countries most fields of activity have become mathematised and the existential meaning of mathematics has changed. With that change has come an important shift in the kinds of cognitive distribution and range that are needed in educational settings if they are to provide the kinds of experiences that will support young people in their development for actions in the wider society. Experience by learners of cognitive activity in the personal planes, as well as operational experiences, are needed to ensure that people have both the capabilities, inclinations and the understanding to interpret machine results critically, to define problems, to decide on or critique the assumed relationships between the relevant variables in a machine based solution strategies, to be creative in using mathematics to solve problems in ways that are personally satisfying and publicly scrutinised.

For parents, industrial leaders and other members of the wider community with life long experiences of hierarchical organisational structures for whom the existential experience of education was 'hard work', 'book work', 'home work', 'training' and 'know how', uncertainty about positive student responses to problem based learning and capability portfolio's is understandable. However, it is also important to understand why are these approaches are favoured by the young. Perhaps they recognise that outside the unchanging schools there are new learning activities. That they will need to keep learning for most of their lives in a society that now changes more. The new forms of learning outside formal institutions which have such an attraction for the young are:

- highly visual and interactive,
- involve working cooperatively with others,
- involve creative mathematisation (even of creative arts),
- provide experiences in information design and representation in many fields
- involve experimenting with multiple roles and in a number of communities of practice,
- often take the form of strategy games, extensive use of domestic computers, networked environments where information and advice can be made available from people in other parts of the world,
- allow a more granular just in time curriculum with multiple paths that is connected to personal purposes and needs,
- clarify the differences between information and the capability and knowledge to make quality transformations of information.

- place a premium on adaptation and change solutions rather than accurate and speedy computation (now machines).

Emerging networked technologies could be used in ways that allow alternative forms of social organisation suitable for mathematical learning and making in personally meaningful ways. Such activities as mentoring by community members, consultation, and cooperative problem solving in groups could be technically supported. However, such learning activities will necessarily involve a review of traditional power relationships in the education industry so that greater agency is allowed to both learners and teachers and more diverse outcomes are acceptable.

However, if traditional hierarchical forms of socialisation and social organisation persist and are accepted then it is likely that 'black box' machines designed by an elite will increasingly use mathematics to determine the virtual context for human experiences with consequences in terms of equity, justice, privacy and personal agency and creativity.

The following questions seem useful for research:

> **What is the nature of mathematical learning, inclinations and capabilities that emerge from experiences of new forms cognitive activity during mathematical activity in communities supported by new interactive technological artefacts?**
>
> **Are particular forms of thinking and learning enhanced by new technological developments? - under what conditions?**
>
> **What kinds of virtual social organisation in educational settings best realised the human potential of new networked, interactive mathetic environments?**

The cultural historical influences on research and development associated with learning mathematics are well known. The impact of government ideologies on educational research and research funding are felt by all researchers and particularly those employed by government institutions. The recent change in funding policy by the National Science Foundation in the United States to research which supports learning and technological development in schools is a case in point. A great deal of research has been done to explore ways of changing educational practice in schools. Commercial pressures are also conservative. For example a large amount of research by BBN

Laboratories to develop exploratory environments in science and mathematics has shown great potential for facilitating new kinds of learning of the kinds discussed above, but has been found very uncomfortable for teachers. The diffusion of Seymour Papert's (1980) vision for LOGO stimulated a great deal of interest and excellent research (eg. Disessa, Hoyles & Noss, 1995) but diffusion after successful projects remains difficult.

## Teaching and Learning in Networked Environments

New networked environments seem likely to compete with traditional forms of education in mathematics. This is likely because children and young people find the new technological environments very attractive (Papert; 1993). A focus for research might be:

**What is a *teacher* and a *learner* of mathematics in these new technological settings?**

There are some beginning clues emerging.

The New Directions in Distance Learning (NDDL) project in Canada has been set up to provide alternative forms of learning for matriculation in remote regions. The project runs in competition with the usual school experiences in a number of fields including mathematics. The success of the early stages of the project was indicated by the fact that in some centres more parents and students were choosing the alternative approach than were choosing the usual school curriculum. The matriculation results of students in the new courses support their choice. In the NDDL project the *local learning facilitator* is not necessarily an expert in mathematics but has a role to support student learning, to facilitate on-line communication with a mentor-teacher who is an expert and other mentors or users of mathematics in the community, and provide technical support. The learning facilitator also supports students in using mathematical materials available on the network. For the learners, the hours are flexible, they can log in at times to suit themselves and only need to be present for audio (or video) conferences with mentors. The communications enhanced, expert supported course is slightly less expensive to run than traditional approaches. One of the reasons for this is that a great deal of work by members of the distance education bureaucracy in preparation and dissemination of hard copy materials is by-passed.

The School Links Project at the University of Sydney has as its major aim the development of a networked, communication enhanced exploratory environment in mathematics which will support learning mathematics in personally meaningful ways and more autonomous approaches to thinking about mathematics. In this case the cyber environment is being designed with a focus on the kind of virtual social context that is being created, the expectations of learners and mentors in the Mathematics Department and the interactive affordances of the technology.

The environment has been specifically designed to support a shift from a focus in implementing and memorising techniques to a deeper approach to learning mathematics as a basis for developing capabilities and inclinations for mathematical interpretation, problem definition, problem solving and evaluation of machine based solutions. It is an exploratory environment where risk taking is encouraged and self evaluation is required. There are game-like activities which are fun, opportunities to create and solve personally meaningful problems and examples of interesting activities. There is information, from experts, about how and why mathematics is used in various fields and how the ideas in one module relate to other mathematics available on the site. Peer consultation and team solutions are also encouraged through the design of the communication functions and the shared spaces at the site.

Reflection on emerging ideas and refinement of initial conclusions is supported by both the interactive mathematical and graphic environment based on Maple, available via an internet plugin, and the additional social features of the context. For example, students can write and edit functions and graph them interactively..

Enduring forms of human learning are associated with domestic activities, personal goals needs and desires that sometimes change throughout a life time. It is planned that this facility will be available to individuals and educational settings where support of this kind for learning mathematics is desired.

## Shifts in Cognitive Distribution and Emerging Capabilities and Inclinations

There are beginning signs that the social conception of a teacher as an implementer of the approved curriculum is changing. In societies where this is occurring there are often anxieties that teachers will be replaced by machines. These anxieties are based on the apparent automation of many teaching roles and much of the kinds of knowledge taught in school. It is therefore important that research investigates:

> **What is the emerging role of a teacher (or learning facilitators) of mathematics in technological settings?**

### What impact does this changing role have on the nature of cognitive activity during learning and the kinds of learning outcomes that occur?

The new learning facilitators will be expert at ensuring that their students learn to learn and are aware of the kinds of experiences that are needed in order to gain different types of knowledge, capabilities and inclinations. Their repertoire of strategies may include;

- direct instruction for those seeking to master operational procedures and set routines.
- support in the use of mathematics for personally meaningful problem solving,
- support in the development of strategies to check the quality of mathematical solutions,
- opportunities for reflective discussions about mathematical ideas and their possible applications,
- support and advice in the interpretation of mathematical information
- instruction on how to evaluate and choose from the multiple offerings available in the form of games, interactive environments, microworlds, mentoring services and course information

In particular, learning facilitators in mathematics will ensure that the full range of cognitive processes are experienced by the learner either in a range of scaffolded situations or through individual and group problem solving experiences which involve choice (cooperation and decision making, choosing), are personally meaningful (links with the personal planes of the learners - deep approaches to learning Marton (1988)), give experience of defining a goal mathematically, interpreting mathematical information, designing or choosing a mathematical strategy to achieve a solution, implementing it and evaluating the results of effort in terms of the original questions asked and goals. Experiences might also include explorations of uses and limitations of various pieces of technology, comparisons of different solution strategies and explaining their own solutions to others.

## References

Crawford, K (in Press) Learning in an organisation, To be published by the University of Sydney Press.

Crawford, K., Vygotskian approaches to human development in the information era, *Educational Studies in Mathematics*, Vol 31, Nos.1-2, September 1996, pp.43-62.

Crawford, K., Cognitive and Social Factors in Problem Solving Behaviour. In the Proceedings of the *Tenth Conference of the International Group for the Psychology of Mathematics Education*, London, July 1986, 415-412.

Crawford, K., Adler, J., Teachers as researchers in mathematics education, in A. Bishop et al. (Eds.), *International Handbook of Mathematics Education*, The Netherlands, Kluwer Academic Publishers, 1996, pp. 101-122.

Crawford, K.; Gordon, S; Nicholas,J. and Prosser, M., Conceptions of Mathematics and How it is Learned: The Perspectives of Students Entering University. *Learning and Instruction*. Volume 4, 1995, pp331-345.

Crawford, K., & Deer, C. E., Do We Practice What we Preach: Putting Policy into Practice in Teacher Education. in the *Journal of South Pacific Association of Teacher Education*,, 21(2),1993, pp 111-121

DiSessa, A., Hoyles C., and Noss, R. (Eds.) *Computers and Exploratory Learning* Springer Verlag, 1995

Marton, F. (1988) Describing and improving learning, in R. Schmeck, (Ed), Learning Strategies and Learning Styles, NY, Plenum Press

Papert, S (1980) Mindstorms, USA, Harvester Wheatsheaf

Papert, S (1993) The Children's Machine, USA, Harvester Wheatsheaf

Vygotsky L.S (1978) *Mind in Society*, Cambridge, USA, Harvard University Press

# ROLES FOR TEACHERS, AND COMPUTERS.

## A contribution to the Plenary Panel: Cognition, Technology and Change

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### Introduction

In this paper I aim to explore just one small corner of the overall topic for this plenary panel: that of the role of the teacher. However, I am going to approach this rather obliquely by looking at roles for the computer. I shall draw on a few incidents from across areas of my current research, which I hope to bring to life more vividly in the panel presentation. I hope that the paper will offer you some stimulus for personal reflection on the impact of technology on your own learning and teaching.

### Potential roles for the computer

> *In many schools today, the phrase 'computer aided instruction' means making the computer teach the child. One might say the* computer is being used to program *the child. In my vision,* the child programs the computer *and, in doing so, both acquires a sense of mastery over a piece of the most modern technology and establishes an intimate contact with some of the deepest ideas ... from mathematics*

If this quotation is not immediately familiar to you, you might like to try, before reading on, to make a guess at *when* it was written - and, or course, by whom. I ask you also to take a moment to think carefully about the images it offers for the role given to the computer, and the child's relationship with it. How do they match your own experiences and expectations, and those of your colleagues and students?

I find it surprising that it is so difficult to recognise that this quotation is at least seventeen years old (it was written by Seymour Papert (1980), in the introduction to *Mindstorms*). What dates it most clearly for me are the word 'instruction' and the reference to programming, both of which are somewhat out of fashion in educational discussion, at least in the U.K. What does not feel at all out of date is the tension expressed between computers being seen primarily as rigid and mechanistic tools for teaching, and as tools for learning. It is also interesting to note that the teacher does not appear. Although the developments in *technology* have been enormous during those seventeen years, ambiguity about the computer's role, and the teacher's role in a computer-classroom, continue to be causes of anxiety for many practitioners.

By taking a narrow focus for this paper, there is a danger of over-simplifying what is in reality a very complex situation, in at least two ways. First, lack of clarity about relative roles of teacher and computer (and, of course, learner) is only one of a long list of factors which affect the extent and quality of the use of computers in mathematics classrooms. Issues to do with access to appropriate hardware and software, curriculum constraints and assessment requirements, attitudes to technology and management issues at both classroom and school level are all extremely significant. Within my own research (of which I shall say more later) it has become clear that even when high levels of access are available, and curriculum pressures relaxed, teachers' confidence in integrating technology within their existing classroom practice remains a key issue (Ainley and Pratt (1995)).

A second over-simplification, which may already be distressing some readers, is that of treating 'technology' or 'the computer' as a single entity. Clearly this is mistaken. Particular pieces of hardware and software will make subtle and not-so-subtle changes to the role which the computer may be given in the classroom. Even the size and physical location of the computer may be significant: think about a desktop machine, with a large, visible monitor, which can only be plugged in one corner of the room, in contrast to a small portable machine which fits comfortably on a table where the screen can be read by only two children.

In order to try to recognise this diversity without getting bogged down in discussion of particulars, I shall focus on possible ways in which the role of the computer may be constructed, and look at the effects these may have on other roles and relationships within the classroom.

**Computer-as-teaching-aid**

It is possible to construct the role of the computer in a mathematics classroom as a teaching-aid, similar in some ways to existing teaching resources. The computer might be used for demonstration (an 'electronic blackboard'), as a source of information (accessing databases), to give pupils access to a range of (simulated) experiences, to provide consolidation and practice of skills in the form of games and exercises. At first glance, in this role the impact of the computer on pedagogic practice may seem relatively unproblematic, because it is performing functions which are familiar to teachers and to learners.

However, a second more considered glance reveals some difficulties. In most classrooms the computer is still a relatively scarce resource, and so it is only possible to give access for restricted periods of time. This may have the effect of isolating work done on the computer from the main work of the class. If only one group of children are able to work on the computer within the classroom, the teacher may be unable to give much attention to their work. For this reason, software which children can use independently is often popular with teachers and with pupils. Indeed one strand of development in educational software is towards packages which are largely teacher-independent. *Integrated Learning Systems* are an extreme example of this trend; not only setting exercises, but also 'marking' these, evaluating and recording progress, and directing the pupil to the next appropriate level of work. I find the title *Integrated Learning System* disturbingly ambiguous: the system may indeed be integrated, but any learning which takes place will be isolated and disjointed.

The use of such software can effectively put the teacher into the role of manager, and perhaps occasional trouble shooter, while the computer's role may be constructed (by pupils at least) as that of teacher. What is more, it is possible to construct an argument for the advantages of the computer-as-teacher over a human teacher. The computer can be infinitely patient; it can simultaneously provide work at an appropriate level for every child, and keep a complete and detailed record of their work. (I shall leave it as an exercise for the reader to find the bugs in this argument.)

Although some teachers may appreciate what such technology can offer in terms of motivation for pupils who thrive on the success they experience, or in terms of the detailed information such a system can provide on pupils' progress, the computer may also feel like an intruder in the classroom. The software designer will have made decisions about both presentation and content which, however well intentioned, are beyond the teacher's control. Although the same is true of other resources, such as textbooks, there is something very powerful about the interactive nature of the computer, and it's inflexibility about the answers it accepts, which creates an impression of 'correctness' some teachers may find intimidating, or frustrating.

**Computer-as-tool**

In contrast to the trend of development described in the previous section, which leads to systems in which the role of the computer may be constructed as teacher, or at least as curriculum-deliverer, many other developments in mathematics education are pointing towards a contrasting role: the computer as a mathematical tool. The increasing use of generic software, often designed primarily for commercial rather than educational uses, offers a different image of the relationships between teacher, learner and computer. Here the teacher's traditional role is not threatened by the technology, since the computer remains passive and neutral, although there may be dramatic implications for change in curriculum terms when, for example, algebraic manipulation can be handed over to the computer.

However, the construction of computer-as-tool is not unproblematic for many teachers who may feel anxious about the need to learn to control and use powerful and sophisticated pieces of software. Particularly when access to computer time is limited, learning the skills of using the software may become seen as something separate; possibly even as a separate *topic* within the curriculum. This is illustrated by Prestage (1996) commenting on the U.K. National Curriculum for Mathematics:

> *... the emphasis in the mathematics document was placed upon using the tools, e.g. use a calculator or use a database, rather than aiding pupils in choosing the appropriate tool for doing mathematics. This was not the case in general, for example, the document did not state 'use a ruler to measure'.*

**Computer-as-tutee**

This may initially seem a strange way in which to construct the computer's role, although it is implicit in the quotation from Papert with which we started. In Papert's discussion of Logo, 'teaching the computer' becomes a natural metaphor for programming. Putting the learner in the role of teacher has been recognised in many areas, from studies of formal peer-tutoring to less structured discussion, as a powerful way of supporting and reinforcing learning.

The computer is different from a human tutee in a number of ways which intensify the learning benefits for its 'teacher'. It has no common sense: it obeys instructions without evaluating or elaborating them, or making assumptions. It forces the 'teacher' to be explicit and precise. So, a child trying to teach the turtle to draw a shape has to give numerical values to distances and angles. The computer is also very patient. It will not criticise, and it will wait all day while many versions of a procedure are tried. But the computer is pedantic: it disciplines the communication with its teacher by only accepting instructions which follow particular conventions. Turns must be given as numbers of degrees to left or right, values can be entered as decimals, but not as fractions. Finally, the computer provides immediate feedback about what it has been taught: the image produced on the screen can be used directly to see how successfully instructions have been given.

**Exploring the metaphor**

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In my own work in the Primary Laptop Project, the children's personal relationships with portable technology are seen as a key element. The children's sense of 'ownership' of their computer (shared in a group of 2 or 3) is encouraged by giving them responsibility for taking care of the machine, for taking it home regularly, and for making decisions about when I how it is used (Pratt and Ainley (in press)). We[1] have used the metaphor of teaching the computer with children working in a number of different computer environments. Exploring the metaphor, both in relation to our work with children and in reflecting on our own mathematical work with the computer, has helped us to think more deeply about what it is that the computer offers to the learner, and to the teacher. I would like to share with you two brief examples from our observations, one involving the use of a spreadsheet, and the other using dynamic geometry software.

The sheep-pen problem

This activity involved children in manipulating a physical model of a sheep-pen built using flexible fencing to form three sides of a rectangle against a wall. They then had to take measurements, and enter these on a spreadsheet they set up to calculate the area of the pen. They could then graph this data to explore the maximum possible area. (For a detailed description of this activity, see Ainley (1996)). At some point during this process most groups realised that they could manage without using the physical model: they could chose a width for the pen, and calculate the corresponding length. At this point, we intervened to encourage the children to encapsulate their method of calculation in a spreadsheet formula, using the metaphor of teaching the computer. The following extract of transcript is typical of these interactions.

*Researcher:* .. *What you are trying to do is to tell the computer how to work the length out, given some width. So if you knew what that width was, you're trying to work that length out* (pointing to the length column.)
*J* *You have to add these together* (pointing vaguely at the length and width column). ... *double it* (pointing to the width).
*S* *How do you double it?* ..
*J* *and then you work out the length.*
*S* *zero point five add zero point five or something*
*J* .. *yeah but they don't know* .. (pointing at width cell)
*J* *I know B eleven,* (typing) *B 11. B11. .... right B 11, add. ... B11 add. oh no. B11 times 2*

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[1] My colleague Dave Pratt is joint director of this project.

*S:* *oh yeah times 2*
*J:* *so then that doubles it , and*
*S:* *add A 11*
*J:* *B11 times 2 add ...*
*S:* *add A11 equals C11*
*J:* *No we need to ... if there's 30 in the ruler right, it's all doubled though, we need to tell it how to work out what's left.*

The task of teaching the computer was familiar to the pupils, and they are able to engage with it immediately. They quickly realised that they needed to express their method in terms of the formal notation of the spreadsheet. Their mental method was to choose a width for the pen (S starts trying to use zero point five as the width), double this, and take that length away from 30, (the total length of the fence). After struggling for some time to express this method clearly, and trying several slightly different verbal formulations, they typed **=30  B11*2** . The computer gave them a message **Bad formula**, and they quickly deleted this formula and typed **=B11*2-30** .

*S:* *... You can't take 30 from ... um*
*J:* *times it by 2 take it from 30*
*S:* *times it by 2 and take it from 30*
They try putting in 13 for the width and get length -4 and area -52.
*J:* *its probably 52*
*S:* *the minus, shouldn't have put the minus in*
*J:* *I don't know*
*J:* *B11 times it by 2 take it from 30 ... but this looks like take away 30, and we don't ... It should have been 4, so its nearly right .*

At this point the children had an efficient method for performing the necessary calculation themselves, but it was not in a form which the spreadsheet would accept. Imagine a different scenario, in which they had reached this point in an activity and the *teacher* required them to express their method in formal mathematical notation. In this case, there seems little purpose for the formal notation, and the teacher's judgements about what is acceptable may seem arbitrary: after all, the teacher *can* make sense of the instruction 'take it away from 30', and the children know that this method works perfectly well.

After the children had struggled to teach the computer their method for some time, and seemed to have become stuck, we decided to intervene to offer them a different way of looking at the physical model they had been working with.

*Researcher:* *Let's think of it in a different way ... Here's our length of fencing, which is 30* (holding up model). *Let's imagine cutting off our two widths. ...*

*J:* *If we start with 30, take away B11 times 2*

They type =30 - **B11 * 2**, fill down the column, and enter values for the width.

*J:* *[ ] We virtually did that, but it was the other way round.*

With their attention on teaching the computer, the children were apparently quite happy to accept that the spreadsheet needed information in an unambiguous, standardised form, and able to make the link between this notation and their more idiosyncratic method.

The drawing-kit

When some groups of children first explored dynamic geometry software[2] they used it spontaneously to make pictures, but as these were essentially static, the children saw no purpose in *construction* rather than placing points and lines by eye. (For a fuller discussion of this see Pratt and Ainley (in press)). In response to this, we designed an activity which we hoped would help the children to see some purpose in using geometric construction. A group was given the task of making a drawing kit which younger children could use to make their own pictures.

After discussing what shapes such a kit might contain, they considered how these shapes had to behave. It was important that each shape could be moved around on the screen, that its size could be altered, and that it could be produced many times. Their aim was not just to draw a single square or wheel, but to teach the computer methods to produce these shapes. In an introductory demonstration of how to construct an equilateral triangle, we also introduced the possibility of making macros.

Understanding the difference between drawing and construction proved to be difficult for most of the children, because initially the visual impression on the screen was much stronger than other considerations. The idea of 'teaching the computer' was a helpful for distinguishing between the two modes of working. With some assistance, one pair managed to re-produce the method they were shown for constructing an equilateral triangle, and stored this as a macro. They then built a macro for a diamond (rhombus) using the first macro twice. They seemed to use a macro within a macro intuitively without questioning that it would work. During this process, the children were talking about 'teaching the computer', 'procedures' and 'the flip-side'; language which suggested that they were making connections between building a macro and writing a Logo procedure.

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[2] The children were using an early version of *Cabri Geometry*. This offered a more limited set of primitives than more recent versions, in which the drawing-kit activity may actually make little sense.

Another group had constructed a circle, placed points by eye onto the circle and joined them to the centre so that it looked like a wheel with four spokes. When they tried to drag a point on the wheel, the image came apart. One boy commented that several parts were not 'stamped on'. This was an important moment in his transition from drawing to construction, as he saw that the computer had not 'learnt' about these points.

**Extending the metaphor**

I believe that these examples indicate ways in which the idea of teaching the computer may be helpful in supporting children's learning of mathematical ideas, and also in helping them to understand something of the *nature* of those ideas. However, in considering the complexity of the interactions with the computer, it seems that the metaphor of computer-as-tutee is too simplistic. The computer is being 'taught' by accepting instructions, but it is doing much more. It is offering feedback in response to those instructions, in ways which offer both validation for correctness, and support for re-formulation. But it is also offering a different, but equally important, form of validation through its insistence on the use of notation which follows mathematical conventions. The computer might be seen as offering a purposeful context for practising the use of such notation (Ainley (1996)).

I have struggled to find a better metaphor for this role for the computer, and my lack of success is perhaps an indication that its role does not have obvious human parallels. The best I have come up with is critical-listener, or perhaps critical-friend. One of the pivotal influences on my own thinking about teaching and learning has been Bruner's (1966) description of the teacher's role.

> *I would like to suggest that what the teacher must be, ... is a day-to-day working model with whom to interact. It is not that the teacher provides a model to* imitate. *Rather it is that the teacher can become part of the student's internal dialogue - somebody whose respect he wants, someone whose standards he wishes to make his own.*

This has always seemed a frighteningly difficult task for any mere human teacher, but perhaps the complementary roles of human teacher and computer-as-critical-listener may make it approachable.

**Ainley J.** (1996) Purposeful Contexts for Formal Notation in a Spreadsheet Environment, *Journal of Mathematical Behavior*, 15(4) 405-422

**Ainley J. & Pratt D.** (1995)Supporting Teacher's and Children's Mathematical Thinking, *Journal of Information Technology in Teacher Education*, vol 4(1) pp 81-92

**Bruner J. S.** (1966) *Toward a Theory of Instruction*, Cambridge Ma.: Harvard University Press

**Papert S.** (1980). *Mindstorms: children, computers and powerful ideas*, Brighton: Harvester Press

**Pratt D. and Ainley J.** (in press) The Design of Activities for the Abstraction of Geometric Construction, *International Journal of Computers for Mathematical Learning*

**Prestage S.** (1996) Teachers' Perceptions of Sequencing and Progression in the Mathematics National Curriculum, in Johnson, D. C. & Millett, A. (Eds), *Implementing the Mathematics National Curriculum*, London: Paul Chapman Publishing

# Some questions on mathematical learning environments

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## Abstract

This is a contribution to the pleanry panel "Cognition, Technology and Change". Among the several questions initially suggested by Kathryn Crawford, I address here two types of questions: transformation of knowledge and the related complexity of the teacher work. This short paper proposes research questions which may be of interest for the next decade for research in mathematics education considering the development of learning and teaching technologies.

## Two postulates

I take as a basic postulate, largely supported by semiotic, that any effort to express an idea results in both a partial expression of it and in the production of unintended meanings. A corollary is that the translation of an idea from a representation system to another inevitably goes with its transformation (a problem is to know how important will be this transformation, for example when translating my ideas from French to "International English"). A large number of research in mathematics education deal with this question, it is also this phenomena which is at the core of the theory of the didactical transposition.

Then, I start from the fact that the design and the implementation of a mathematical software introduce a new source of transformation of mathematical knowledge. Indeed, these transformations may well come from the beliefs and conceptions of the designers, but I restrict myself here to the specificities and constraints of the representation systems both at the interface and "inside" the computer.

A second postulate which I take is that the nature of the meanings constructed is not only a property of the individual learner, but also of the means of interaction with his or her environment, the nature of the feedback he or she gets, and the nature of the constraints on these interactions. There is also a large number of research which

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support this postulate, as well as several attempts to provide it a theoretical rationale.

Considering the questions raised by Kathryn Crawford, my point is that the nature of the mathematical knowledge that emerges from the interaction with mathematical software may depend in an important way from these transformations.

I will consider this question, as well as the question Kathryn Crawford raised on the role of the mathematics teacher, under the light of the distance teaching/learning project TéléCabri which I currently manage. This project, because of the specific constraints it imposes on interactions, offers a new opportunity to explore the general question of "Cognition, Technology and Change".

## The meaning of the screen

### *A short story*

Cabri-géomètre, the dynamic geometry software, allows to draw a point on a segment without any other constraint than being one of the points of the segment, lets us call it an "any-point" of the segment (to keep the idea that it is randomly choosen).

When one extremity of the segment is dragged on the screen, one expects the any-point to move, but in which way? A decision must be taken about the behaviour of this any-point. One may suggest to express what might happen in a paper-and-pencil environment: to choose randomly a new any-point for each new position of the extremity of the segment . But on the computer screen the any-point would "jump" from place to place, provoking astonishment since users probably expect the drawing to evolve smoothly. This smooth motion is obtained in the case of Cabri-géomètre by constraining the any-point to always divide the segment according to the same ratio. The consequence is that, so to say, this point is no longer an any-point: when one extremity of the segment moves while staying on a given line, the trajectory of the any-point is an homothetic line.

The decision taken by Cabri-géomètre designers might be the object of an endless discussion since in any case such decisions *must* be taken, and any decision will have its side effect. Whatever the decision is, the essential issue is to characterize its effects and possible consequences.

Direct manipulation of geometrical representations on a screen may turn the intended geometrical microworld into a mechanical microworld for which geometry gives efficient means of modelling and control. But to some extent Cabri-géomètre gives us more than just Euclidean geometry.

*Two research questions*

Again, the question is not to suppress effects, but to be able to say in details what they are. Could we characterize the domain of validity of the chosen representations, and as a consequence the domain of validity of the educational software itself?

The didactical importance of answering this question is due to the fact that the student, as a learner, does not have the means to decide of the validity of what he or she oberves on the screen.

Let us take the case of functions and of the use of graphic calculators. Its is largely argued that the screen display of the graph of a function allows to play between the graphical setting and the algebraic setting, and that it gives access to the richness of visualisation of functions properties. But is must not be forgotten that the observed phenomena are the result of sophisticated computations and programming negociations of the constraints of the interface, and so the display of a function could violate the mathematical model. Learners have better not to believe the screen (or their eyes) but to put under question the validity of what they observe. Discussing the case of Computer Algebra Systems, Hillel (1993) suggests even the possible need for a specific vocabulary to refer to the software specificities ("window", "practical graph", "visual slope", etc.), and he proposes a new mathematical activity: "window shoping".

When working with a well validated software, at the core of its (in general implicit) domain of validity, where What You See Is What Mathematics Expects, observing the screen is of a high heuristical value. When working at the boarder line or whith a non certified software or in a domain where one is not sure of being in good control of what is displayed, then what the screen offers must be scrutinized.

Since it is not possible to find a solution avoiding bias between representations and what they intend to represent, given a software, I suggest that a research question is to delineate its *epistemological domain of validity* (Balacheff and Sutherland 1994). To tell it briefly: A domain in which it opens a valid interaction with the represented mathematical objects.

One may ask the question of why it is a research question. There is a lot of material telling what a software is about (from the description of the specifications to the users manual). The answer is that the complexity of programming and of the technology is such that it is not possible to guaranty that the technological artefact will behave as predicted, and the methodology to model it is even not clear.

One way could be to analyse and understand the *computational transposition* (Balacheff 1993) which leads to the adaptation of knowledge representations to the

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requirements of computable models. This call for a close collaboration between computer-scientists and researchers in mathematics education.

The stake of these questions is the mathematical validation of the software we use. But let us now turn to the other side of the question raised by Kathryn Crawford: What does the student learn as the result of the interaction with the machine? Which problem does it raise for the teacher ?

## Sharing meaning through the screen

*A not so short story*

A student and a teacher communicate through TéléCabri, a distance teaching/learning platform allowing point-to-point computer-based communications including visiocommunication and the sharing of Cabri-géomètre between the distant partners. The basic scenario alternates phases of student autonomous work and, at will, phases of interactions at a distance with a teacher (Balacheff and Soury-Lavergne 1995). In the present short story, the student was coping with the following problem (I leave the reader to Cabri-construct the corresponding figure):

> Construct a triangle ABC. Construct a point P and its symmetrical point P1 about A. Construct the symmetrical point P2 of P about B, construct the symmetrical point P3 of P about C. Move P. What can be said about the figure when P3 and P are coincident? Construct the point I, the midpoint of [PP3]. What can be said about the point I when P is moved? Explain.
>
> From Capponi (1995) Cabri-classe, sheet 4-10.

The student calls the teacher to submit his answer to the first question and to express his concern with the second question. The point I don't move, but so what?... The teacher encourages him to focus on the parallelogram ABCI that he had *noticed* spontaneously. The student proves that the ABCI is a parallelogramme. At this stage, from the point of view of geometry (and of the teacher), the reason why I remains immobile while P is manipulated, is obvious. The teacher provides several hints to the student who does not see the "obviousness":

> " *Hey! What does is mean that when one moves P, I does not move? It tells you that I is how?*" [prot.113]
>
> "*But if it does not move when you move P. That tells you what? I, the point I, you have told me that it moved according to which points?*" [prot.139]

> *"The others, they do not move. You see what I mean? Then how could you define the point I, finally, without using the points P, P1, P2, P3?"* [prot.143.]

The teacher makes every effort in order to obtain an appropriate answer from the student, but the student witnesses the failure of these efforts: *" I do not see from where to start"* [prot.150]. The current episode stops at this point, the teacher leaving the student to return to an autonomous phase of work. It is during the next communication that the teacher tells to the student the reasons, provoking a genuine ha-ha effect...

Two different tasks have been undertaken in the course of this interaction: (i) to show that ABCI is a parallelogram; (ii) is to show that the location of I does not depend on P, P1, P2 and P3.

The student engaged in the first task (i), but without establishing a link with the second (ii). It is this gap, which constituted the stake of the interaction and which turned into an obstacle. In order to explain the immobility of I, the teacher had to obtain from the student the construction of a link between two worlds:

- the first, a theoretical world , which is that of geometry, and
- the second, a mechanical world, which is that of the domain of interface of Cabri-géomètre.

Only this link can turn the observed *fact* of the *immobility of I* into a *phenomenon*, that is the property of *invariance of I*. The teacher's interventions are a kind of maïeutic and one may wonder why she does not simply give the full explanation of which she has gathered all the elements. The answer is probably in the didactical contract which functions here as a paradoxical injunction: the more precisely she would tell to the pupil what he has to do, the more she risks provoking the disappearance of the expected learning (Brousseau 1997, p.66).

The passage from mechanisms to geometry is not obvious, *it is a process of modelling*. The animation of figures, which could be seen as dynamically providing explanation given by the machine, could turn into an obstacle linked to the nature of these representations (articulated mechanisms) with respect to their object (geometry). In order to overcome this obstacle, the relationship between the dynamic representation system and the represented knowledge must be considered by the teacher as an object of knowledge.

*Several research questions*

Teachers will not be fully able to insert the new technology into their daily practice, if they are not well informed on all the aspects which could determine its place and its precise role in a didactical process. This call for research on the didactical specificities of situations involving computer-based learning environment. Especially, how could we conceive of the communication between the teacher and the machine about the learning process. Could the machine be able to handle and produce relevant didactical information about the teaching process, in order to interact and cooperate properly with the teacher.

Considering the issue of teaching at a distance this question is crucial. We have to understand the complexity of the tutoring task in the context of telepresence and to specify of the tools needed by the human distant tutor. If the student has benefited from considerable research aimed at a better adaptation of the machine to his or her needs, this is not at all the case for the teacher.

This is not a problem to be left to computer-scientists, the stake of the coming development of mathematical learning environment is our capacity to develop more formal models of learning and teaching in order ot bridge computer-science and education.

## References

Balacheff, N. (1993) Artificial Intelligence and Real Teaching, in Keitel C. et al. (eds.) *Learning Through Computers: Mathematics and Educational Tedchnology*, Berlin: Springer Verlag.

Balacheff, N., Sophie Soury-Lavergne (1996) Explication et préceptorat, à propos d'une étude de cas dans TéléCabri. In: Joab M. (ed.) *Explication' 96*. (pp.343-356). Sophia Antipolis: INRIA.

Balacheff, N. & Sutherland, R. (1994) Epistemological Domain of Validity of Microworlds : The Case of Logo and Cabri-géomètre, in Lewis R & Mendelsohn P (ed.) *Lessons from Learning*, IFIP Conference TC3WG3.3 North Holland, pp 137-150.

Brousseau, G. (1997) *Theory of Didactical Situations*. Dordrecht: Kluwer Academic Publishers.

Hillel, J. (1993) Computer Algebra Systems as Cognitive Technologies: Implication for the Practice of Mathematics Education, in Keitel C. and Ruthven K. (eds.), *Learning Through Computers: Mathematics and Educational Tedchnology*, Berlin: Springer Verlag.

# Deepening the Impact of Technology Beyond Assistance with Traditional Formalisms In Order to Democratize Access to Ideas Underlying Calculus

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## ABSTRACT

*This paper addresses the question of how we might exploit interactive technologies to democratize access to ideas that have historically required extensive algebraic prerequisites. Illustrations will be drawn from work in the authors' ongoing SimCalc Project, which builds and tests software simulations and related visualization tools intended to render more learnable the ideas underlying calculus beginning in the early grades. We will reflect on how such technologies can change the experienced nature of the subject matter by tapping more deeply into students' cognitive, linguistic and kinesthetic resources. Substantial reorganizations are possible of curricula that have been taken as given for centuries.*

## Introduction

Inherited approaches to Calculus are the product of several centuries of evolution. The curriculum and texts developed at the hands of masters in the 17th and 18th centuries. Many basic curricular structures set down in textbooks by L'Hopital, the Bernoulli's, Euler, and their contemporaries, have remained largely invariant through the 20th century. This is not merely a matter of inertia, because these structures served traditional purposes and populations extremely well. Indeed, this basic intellectual material is at the foundation of our civilization's scientific and technological infrastructure that we now regard as natural as the earth and sky. While its educational forms evolved into an almost sacred academic tradition (MacLane, 1984), the ambient societies, the nature of education, and the relations between education and the larger society, including and especially in the United States, changed and continue to change profoundly. In the United States these changes, especially technologically-driven changes, and resulting educational ill-fit with traditional forms, have led to a major university-centered "Calculus Reform Movement" (Tucker, 1990). However, these reforms have two basic characteristics that our current work does not share: (1) they are university-centered, intending to reform the teaching of calculus at the university level, and (2) they focus on the use of interactive technologies to facilitate the use of traditional notation systems, both to manipulate within systems as well as to link between representational systems, especially numeric, graphical and algebraic systems (the traditional "Big Three").

In contrast, first, our approach treats the underlying ideas of calculus: variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations as the focus of school mathematics beginning in the early grades and rooted in children's everyday experience, especially their kinesthetic experience. And second, we use the computational medium both to create new notations and actions upon these as well as to bring phenomena into the center of the educational activity - by means of directly student-controllable simulations and the importing of physical data into the computational environment. These strategies are not intended to eliminate the need for eventual use of formal notations for some students, and perhaps some formal notations for all students. Rather, they are intended to provide a substantial mathematical experience for the 90% of students in the US who do not have access to the Mathematics of Change & Variation (MCV), including the ideas underlying Calculus, and provide a conceptual foundation for the approximately 10% of the population who need to learn more formal Calculus. Finally, these strategies are intended to lead into the mathematics of dynamical systems and its use in modeling nonlinear phenomena of the sort that is growing dramatically in importance as we move into the 21st century (Cohen & Stewart, 1994; Hall, 1992; Stewart, 1990).

The remainder of this paper will introduce the SimCalc Project approach to the MCV as reflected in its first software product, MathWorlds. Empirical research reports are in press (Roschelle, Kaput, & Stroup, in press) and in preparation.

## Design Goals Leading to *SimCalc MathWorlds*

We sought to ground the design of learning activities in a thorough understanding of the experiences, resources, and skills students can bring to the MCV. We initially examined attempts by the Scholastics to mathematize change before algebra was available (Claggett, 1968; Kaput, 1994), and took into account the large literature on students' difficulties with kinematics (McDermott, et al., 1987) and graphs (Leinhardt, et al. 1990). Our aim was to build the ideas to which the more formal algebraic notations conceptually refer, the ideas that they are "about." These key underlying ideas of rate of change, accumulation, the connections between variable rates and accumulation, and approximation, all have forms sensible to young students from diverse populations. We work with students ranging in age from 6 and 7 years to university students. Following the historical lead and recognizing that the language and metaphors of motion are used quite generally to describe change and variation, we focused (although not exclusively) on mathematizing linear motion, particularly by controlling motion simulations in familiar or fanciful situations: elevators, people walking or dancing, cars, duckies on a pond, boats in a river, space-vehicles, and so on (see Figure A).

Research at TERC and elsewhere (e.g., the Shell Centre) has uncovered the important roles of physical motion in understanding mathematical representations (Nemirovsky et al., in press; Nemirovsky & Noble, in press). In studying their own movement, students confront subtle relations among their kinesthetic sense of motion, interpretations of other objects' motions, and graphical, tabular and even algebraic notations.

Our starting criteria were to begin with students' intuitive experience with speed and motion, minimize *computational* complexity, and yet maintain sufficient variation to avoid the conceptual degeneracy of constant velocity and linear functions (Stroup, 1996). These criteria led to extensive use of piecewise constant velocity functions as shown in Figures (A.1-2). Furthermore, we wanted to support direct graphical manipulation of these velocity functions - after all, defining and manipulating piecewise constant functions algebraically is a very cumbersome process, and the vertical arrow in A.2 indicates a dragging action to change the height of the velocity graph segment to which it is attached.

Moreover, in a reversal and complement to Microcomputer-based labs (MBL), TERC developed the concept of Lines Become Motion (LBM) in which graphical representations on a computer control physical devices. Their studies of functions and derivatives in MBL (with body motion, air and water flow) led us to realize the need for students to use symbols, especially graphs, to control phenomena, not just to interpret them. These findings support the inclusion of MBL capabilities in MathWorlds. An illustration is provided in Fig. A.5, discussed below.

Yet another major source of design consideration supporting piecewise defined functions, is also based in the work of our colleagues at TERC, who found that children **spontaneously** engage in "interval analysis" to understand the graphical behavior of a complex mathematical function. Without explicit instruction students parse a graph into intervals based on their understanding of the events that the graph represents (Nemirovsky, 1994; Monk & Nemirovsky, 1994), where the intervals correspond to identifiable, separable sub-events. Within this framework students understand curved pieces of graphs as signifying behaviors of objects or properties of events, rather than as sets of ordered pairs. They also readily constructed more flexible and richer schemes as they made sense of increasingly complex situations and constructed rich mathematical narratives that tell the story of a graph over time (Nemirovsky, 1996). These well-documented student resources directly influenced our focus on piecewise defined and editable functions.

## An Overview of *SimCalc MathWorlds*

MathWorlds provides a very rich set of tools in a flexible environment reflecting our component software architecture (Roschelle & Kaput, 1996). Only perhaps 5% of the features will be discussed below, due to space limits.

Constructing Motions: In MathWorlds, by dragging an appropriate icon from the vertical toolbar (shown on the right side of the screen in Fig. A) the student or teacher can easily construct a function by concatenating segments of velocity or acceleration which specify a function over a specified duration. In a velocity graph, these functions appear as discrete steps (constant velocity) or rising or falling lines (constant acceleration). MathWorlds provides direct click-and-drag editing of any segment. For example, a user can drag the top of a rectangular velocity segment as in Fig. A.1 higher to make a faster velocity. Or a user can drag the right edge of the rectangular segment to the right to give the motion a longer duration. Students can also construct a function (or extend an existing one) by dragging additional segments into the graph. Thus operations on the representation have clear and simple qualitative interpretations. For example, Fig. A.1 shows a velocity graph that controls the elevator on the left side of the screen, which will travel at 3 floors/sec for 2 seconds. As indicated in Fig. A.4. a linear or parabolic function can be constructed using a single piecewise linear segment (where, say, a velocity segment can have zero slope, yielding a linear position graph). In addition, MathWorlds can accept standard input of exponential and periodic functions as well as direct drag-based graphical editing of such functions.

Determining Mean Values: Fig. A.2 shows two velocity graphs, each controlling one of the two elevators (the graphs are color-coded to match the elevator that they control). The downward-stepping, but positive, velocity graph typically leads to a conflict with expectations, because most students associate it with a downward motion. However, by constructing it and observing the associated motion (often with many deliberate repetitions and variations), the conflicts lead to new and deeper understandings of both graphs and motion. The second graph in Fig. A.2 provides constant velocity and

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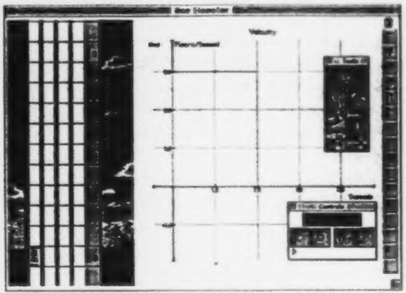
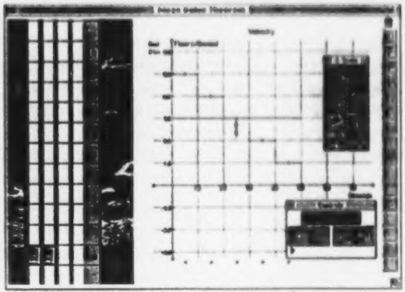
is shown in the midst of being adjusted to satisfy the constraint of "getting to the same floor at exactly the same time." This amounts to constructing the mean value of, or the average velocity of the other elevator which has the variable velocity. This in turn reduces to finding a constant velocity segment with the same area under it as does the staircase graph. In this case the total area is 15 and the number of seconds of the "trip" is 5, so the mean value is a whole number, namely, 3. It is possible to configure MathWorlds so that all segment endpoints have whole number coordinates - this is denoted and experienced as "snap-to-grid" because, as dragging occurs, the pointer jumps from point to point in the discrete coordinate system. Note that if we had provided 6 steps instead of 5, the constraint of getting to the same floor at exactly the same time (from the same starting-floor) could not be satisfied with a whole number constant velocity, hence could not be reached with "snap-to-grid" turned on.

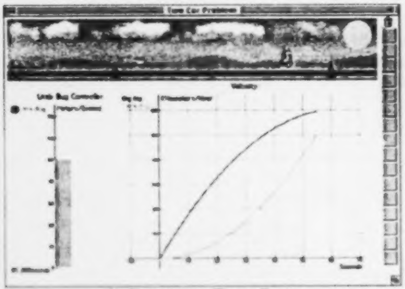
Functions Defined by Sampled Data: MathWorlds provides a range of other function types to complement piecewise linear functions. A "sampled" function type supports continuously varying positions, velocities, or accelerations. These data points can be entered directly with the mouse (by sketching the desired curve, ala Stroup, 1996), from Microcomputer-based Laboratory (MBL) data collection gear (Mokros & Tinker, 1987; Thornton, 1992), or by importing mathematical data from another software package such as Function Probe (Confrey, 1991). Motion can also be controlled in real-time through the use of a mouse-driven "velocity-meter" or "accelerator-meter." The most typical scenario is pictured in Fig. A.3, where one vehicle has its motion given in advance and the second vehicle is controlled by one of the meters in real time. The task might involve following behind the given vehicle at a specified distance, for example. Furthermore, the given motion might be described via a position vs time graph while the student's feedback on the car that she is controlling might be in terms of a velocity vs time graph. Here, in Fig. A.3 by using the controller on the left to drive the "VW Bug"with a concave up velocity graph, the student is enacting a typically confusing situation involving two cars that begin side-by-side but where one has a concave up velocity graph and the other is to have a concave down velocity graph which crosses the first at a certain point in time. Well-documented student expectations assume that the cars will be adjacent when their velocity graphs are adjacent. By "driving" in such situations and many variations on them, the students come to see not only that this adjacency is not the case, but could never be the case. Fig. A.5 illustrates how a sampled function from a motion sensor can drive an actor in a simulation - the "Froggie Dude" character in the bottom of the picture. A student has created a motion physically by moving in front of the motion-sensor, this data has been uploaded to MathWorlds, and attached to Froggie Dude. Then the student created a series of "Clown" characters and synthetic motions for each using piecewise linear functions. In effect, the student is "leading his own Clown Parade." Note that Fig. A.5 shows the parade in progress, so only the first part of the graphs is revealed.



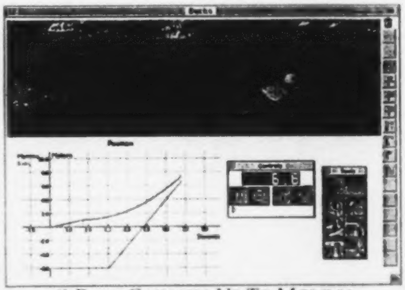
1) AN ELEVATOR AT 3 FLOORS/SEC FOR 2 SECONDS



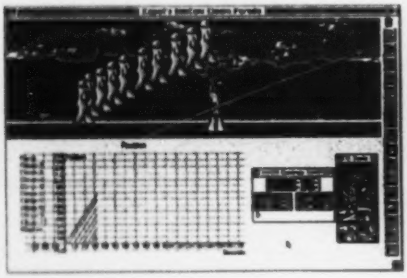
2) THE MEAN VALUE OF THE STAIRCASE



3) DRIVING TOY CARS



4) BABY CATCHES UP TO MOMMA



5) MBL DUDE LEADS A CLOWN PARADE

6) MIXING KINESTHETIC EXPERIENCE WITH SIMULATIONS

Balancing Conceptual and Phenomenological Richness with Computational Tractability: In terms of children's resources, piecewise linear (or piecewise constant) velocity segments provide a primitive object that can draw effectively upon pre-existing knowledge and skills. For example, middle school students can learn to predict position from a velocity graph by using two skills that they have already developed: counting and area multiplication. The velocity graph in Fig. A.1. is drawn

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against a grid (as are all graphs), which enables students to compute accumulated position by counting grid squares. Indeed, they quickly determine shortcuts based on an area model of multiplication - here the question is, "How far will the elevator travel?" In this case, other knowledge based in natural language may be brought to bear, via locutions such as "Well. since it's going at 3 floors per second for two seconds, it will go six floors." Importantly, there is a direct route from these starting points to the powerful and general idea of integration based on the area under the graph of a function, and usual approximations of the area using rectangles (which are nothing more than our piecewise constant rate-functions). As noted in (Roschelle, Kaput & Stroup, in press), students readily extend their counting skills to deal with the non-constant linear velocity case by counting half squares. Our approach differs from the traditional algebraic approach: (1) in the way we respond to the need for computational tractability, and (2) the greater value we place upon experiencing phenomena — we put phenomena at the referential center of the learning environment. The starting point in the algebraic approach is governed by what is computationally simplest in that algebraic universe - the family of polynomial functions, which in turn leads to linear and quadratic functions as the inevitable starting point for computing derivatives and integrals symbolically. Hence computational tractability drives the algebraic approach in the direction of simple, mathematical forms. Computational tractability in the graphically defined and manipulated universe pushes in a different direction - towards piecewise linearity which affords semantic complexity sufficient to support generalization without sacrificing computational tractability (Duckworth, 1991).

Putting Phenomena At the Center: The second major departure from traditional practice, putting phenomena at the center of the enterprise, is partially served by the graphical approach to piecewise linear functions, which allows richer relations with students' experience of motion. Consider the problem of defining a function that represents the motion of an elevator that will pick up and drop off passengers in a building. While such a function is very difficult to formulate algebraically, it is relatively easy to directly drag hotspots on piecewise linear velocity segments to create an appropriate function. Similarly, defining motion-functions for two characters who are dancing would be extremely cumbersome to do algebraically - cumbersome for younger students in entirely unproductive ways. (Exercise: Write out an algebraic description of the position functions driving Momma and Baby Duckies depicted in Fig. A.4). Equally important to drawing upon children's resources is providing opportunities to make necessary distinctions in places where prior knowledge may be poorly differentiated. A classic example is the distinction between slowing down and moving downward (between "going down and slowing down") forced by the step-graph in Figure A.2 (Moschovich, 1996). More generally, children have great difficulty distinguishing "how much" from "how fast," (Stroup, 1996).

## Conclusions and Next Steps

Prior work indicates that student-manipulable motion simulations, coupled with real data based in students' physical experience, can serve not only the learning of the mathematics of change & variation, including the ideas underlying calculus, but it can also serve to contextualize and organize much other mathematical content in ways that yield the efficiency necessary to provide curricular space for all the new mathematics that students will need in the coming century. While solving problems involving, say determining the mean value of a variable velocity motion, students are also performing richly contextualized computations, relating these to areas and other geometric considerations, finding approximations, dealing with signed numbers, etc. Current work is addressing the problem of formalization - how to integrate the understandings developed in graphical and informal settings with understandings mediated by formal algebraic symbol systems, especially where the algebraic knowledge and the MCV understandings symbiotically co-evolve. We are also examining how the deep yet direct drag/drop configurability afforded by new interfaces can be effectively used by teachers to support their own objectives and tastes - it is a straightforward matter to control the tools and views available to the student, to provide scripted scaffolding, and even connections to other systems such as the Internet (Kaput & Roschelle, 1996). Finally, we are working closely with private sector interests to link computers and hand-held devices fluently in classroom networks, to change their relation from competing to cooperating.

## References

MathWorlds, along with other articles and materials, can be downloaded from the SimCalc web site, http://www.simcalc.umassd.edu/

Claggett, M. (1968) *Nicole Oresme and the medieval geometry of qualities and motions, Chapters I & II.* Madison, WI: University of Wisconsin Press.

Cohen, J. & I. Stewart, (1994) *The collapse of chaos: Discovering simplicity in a complex world.* New York: Viking Books.

Confrey, J. (1991). Function Probe [software]. Ithaca, NY: Department of Education, Cornell University.

Duckworth, E. (1991). Twenty-four, forty-two, and I love you: Keeping it complex. *Harvard Educational Review, 61*(1), 1-24.

Hall, N. (1992) Exploring chaos: A guide to the new science of disorder. New York: Norton.

Kaput, J. (1994.) Democratizing access to calculus: New routes using old roots. In A. Schoenfeld, (Ed.), *Mathematical thinking and problem solving.* Hillsdale, NJ: Erlbaum.

Kaput, J. & Roschelle, J. (1996). Connecting the Connectivity and the Component Revolutions to Deep Curriculum Reform (http://www.ed.gov/Technology/Futures/kaput.html). Washington, DC: Department of Education.

Leinhardt, G., O. Zaslavsky and M. Stein, (1990) Functions, graphs, and graphing: Tasks, learning, and teaching, *Review of Educational Research*, *60*, 1-64.

McDermott, L., M. Rosenquist and E. v. Zee, (1987) Student difficulties in connecting graphs and physics: Examples from kinematics, *Amer. J. Physics*, *55*, 503-513.

Mokros, J.R. & Tinker, R.F. (1987). The impact of microcomputer-based labs on children's ability to interpret graphs. *Journal of research in science teaching, 24*(4), 369-383.

Monk, S.; Nemirovsky, R. (1994) The Case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, J. Kaput & A. Schoenfeld (Eds.), *Research in Collegiate Mathematics Education,* 4,139-168.

Moschovich, J. N. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *Journal of the Learning Sciences, 5*(3), 239-277.

Nemirovsky, R. (1996) Mathematical Narratives. In N. Bednarz, C. Kieran, & L. Lee (Eds.), Approaches to algebra: Perspectives for research and teaching. Dordrecht, The Netherlands: Kluwer Academic Publishers. pp. 197-223

Nemirovsky, R. (1994) On Ways of Symbolizing: The Case of Laura and Velocity Sign. *The Journal of Mathematical Behavior. 13*, 389-422

Nemirovsky, R., Noble, T. (forthcoming) Mathematical visualization and the place where we live. *Educational Studies of Mathematics.*

Nemirovsky, R., Tierney, C., Wright, T. (forthcoming) Body Motion and Graphing. *Cognition and Instruction.*

Roschelle, J. (1992). Learning by collaborating: Convergent conceptual change. Journal of the Learning Sciences, 2(3), 235-276.

Roschelle, J. & Kaput, J. (1996). Educational software architecture and systemic impact: The promise of component software. *Journal of Educational Computing Research, 14*(3), 217-228.

Roschelle, J., Kaput, J, & DeLaura, R. (1996). Scriptable applications: Implementing open architectures in learning technology. In P. Carlson & F. Makedon (Eds.), *Proceedings of Ed-Media 96 - World Conference on Educational Multimedia and Hypermedia*, American Association of Computers. Charlottesville, VA: AACE. 599-604.

Roschelle, J., Kaput, J, & Stroup, W. SimCalc: Accelerating Students' Engagement with the Mathematics of Change. In M. Jacobson & R. Kozma (Eds.) *Educational technology and mathematics and science for the 21st century.* Hillsdale, NJ: Erlbaum.

Stewart, I. (1990) Change. In L. Steen (Ed.), *On the Shoulders of Giants: New approaches to numeracy*, , National Academy Press, Washington, DC, pp. 183-219.

Stroup, W. (1996). Embodying a nominalist constructivism: Making graphical sense of learning the calculus of how much and how fast. Unpublished doctoral dissertation. Cambridge, MA: Harvard University.

Thornton, R. (1992). Enhancing and evaluating students' learning of motion concepts. In A. Tiberghien & H. Mandl (Eds.), *Physics and learning environments* [NATO Science Series]. New York: Springer-Verlag.

Tucker, T. (1990). *Priming the calculus pump: Innovations and resources. Washington*, DC: MAA.

# RESEARCH FORA

**Research Forum**

## ELEMENTARY NUMERICAL THINKING

| | |
|---|---|
| *Coordinator* | Erna Yackel |
| *Presentation* | The nature of the object as an integral component of numerical processes<br>by Eddie Gray, Demetra Pitta and David Tall |
| *Reactors* | Robert B. Davis<br>Dagmar Neuman |

# The Nature of the Object as an Integral Component of Numerical Processes[1]

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*This paper presents an outline of research studies indicating the existence of qualitatively different thinking in elementary number development, drawing on empirical evidence obtained over the last ten years. Evidence first signalled qualitative differences in numerical processing (Gray, 1991) which was seminal in the development of the notion of procept (Gray & Tall, 1994). More recent studies of the role of imagery in elementary number processing (Pitta & Gray, 1997) indicate that qualitatively different outcomes may arise in the abstraction of numerical concepts from numerical processes because children concentrate on different objects or different aspects of objects which are integral components of numerical processes.*

## INTRODUCTION

The notion that numerical concepts are formed from actions with physical objects underpins the conceived cognitive development of simple arithmetic (see, for example, Piaget, 1965; Steffe, et al., 1983; Kamii, 1985; Gray & Tall, 1994). These conceptions share common ground. The properties by which the physical objects are described and classified need to be ignored; and attention is focused on the *actions* on the objects which have the potential to create an 'object of the mind', which has new properties associated with new classifications and new relationships. For some there may be a cognitive shift from concrete to abstract in which the concept of number becomes conceived as a construct that can be manipulated in the mind. For others, however, meaning remains at an enactive level; elementary arithmetic remains a matter of performing or representing an action.

The focus of our work has been to consider this difference in thinking and its consequences. What is it that children are doing differently and why? Using rich empirical evidence we develop a cognitive theory which may account for these differences. Our research paradigm has focused on extremes of mathematical achievement and examined children's interpretations of arithmetical symbolism and the associated imagery. Our conclusions highlight the influence of different cognitive styles influencing the nature the object which is an integral component of children's numerical processing. We suggest that such differences effect the cognitive shift from concrete to abstract thought and will have consequences on children's numerical development.

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[1] Our grateful thanks are extended to the four anonymous reviewers of the PME review panel whose detailed and perceptive comments have contributed to the form of the current paper.

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## A Theoretical Framework

Piaget (1973, p. 80) believed that the growth of numerical knowledge in the child stemmed, "not from the physical properties of particular objects but from the actual actions carried out by the child on the objects". He wrote of how the coordination of actions became mental operations—"actions which could be internalised" (Piaget, 1971, p. 21)—and suggested that "actions or operations become thematised objects of thought or assimilation" (Piaget, 1985, p. 49). The formation and meaning of knowledge, within the context of learning as well as in mathematics, stemmed from active thinking and operating on the environment.

Substantial interest in the cognitive development of mathematics has focused on the relationship between dynamic actions and conceptual entities. For some, grammatical metaphors sharpened the subtle changes that form the basis for numerical constructs. Dienes (1960) described how a predicate (or action) becomes the subject of a further predicate which may in turn become the subject of another and so on. The qualitative benefit from making predicates the servant rather than the master of thought were clear:

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians. (Dienes, 1960, p. 21)

Using a similar analogy Davis (1984) signalled the qualitative changes associated with actions becoming objects of thought.

> The procedure, formerly only a thing to be done—a verb—has now become an object of scrutiny and analysis; it is now, in this sense, a noun. (Davis, 1984, p. 30)

These distinctions, together with theories accounting for the transformation of processes into concepts have helped to shift attention from *doing* mathematics to *knowing* mathematics. The way in which dynamic actions become conceptual entities has been variously described as "interiorisation" (Beth & Piaget, 1966), "encapsulation" (Dubinsky, 1991), or "reification" (Sfard, 1991). Dubinsky and his colleagues (Cottrill et al., 1996) formulate the encapsulation as part of the APOS theory (action-process-object-schema), in which actions become repeatable as processes which are then encapsulated into objects to later become part of a mental schema. Sfard also indicates the cognitive shift as a three phase process: *interiorisation* of the process, then *condensation* as a squeezing of the sequence of operations into a whole, then *reification*—a qualitative change manifested by the ontological shift from *operational* thinking (focusing on mathematical processes) to *structural* thinking (focusing on properties of, and relationships between, mathematical objects).

Gray & Tall (1994) focused on the role of mathematical symbolism representing either a **pro**cess to *do* or a con**cept** to *know*. To emphasise this dual meaning the term **procept** was introduced. Procepts start as simple structures and grow in interiority with the cognitive growth of the child. The word "concept" rather than "object" was used because terms such as "number concept" or "fraction concept" are more common in ordinary language than "number object" or "fraction object". Furthermore, the term is used in a manner related to the "concept image" consisting of "all of the mental pictures and associated properties and processes" related to the concept in the mind of the individual (Tall & Vinner, 1981, p. 152). In this sense there is no claim that there is a "thing" called "a mental object" in the mind.

Instead a symbol is used which can be *spoken, heard, written* and *seen*, which is capable of evoking appropriate processes to carry out necessary manipulations in the mind of the individual and which can be communicated to share with others.

Theories which refer to the cognitive shift from process to object are process driven, but they form an important backdrop for the theory of procepts. Indeed Anna Sfard's notion of duality (Sfard, 1991) and discussions with her in 1989 were important in its early development. Procepts are dynamic and generative—"things" that are the source of great flexibility and power. The problem in the cognitive context is to identify why some children implicitly seem to recognise this fact but others do not.

## A Focus on Elementary Arithmetic

'Encapsulation' theories—and here the one word is used as a matter of convenience—have intrinsic differences but also share common ground in attempting to account for process/object links. Notions such as 'interiorisation' or 'repeatable actions' may lead to quantifiable differences in procedure but not qualitative differences in thinking. Such a distinction is implicit in the finely-grained analysis of counting units of Steffe et al. (1983). Decreasing dependence on *perceptual* material permits children to eventually count *figural* representations of perceptual material: the counting process continues in the absence of the actual items. *Motor acts*, such as pointing, nodding and grasping, that accompany the counting process, can be taken as further substitute units for perceptual items. Dependence on these three forms of unit is further reduced by the realisation that the utterance of a number word, the verbal unit, can be taken as a substitute for countable items that could have been coordinated with the uttered number sequence. However, these changes though quantifiably different, are qualitatively similar—each procedure is an analogue of a fundamental counting process. The concept of unit becomes wholly abstract when the child no longer needs any material to create countable items nor is it necessary to use any counting process.

## The Empirical Evidence

Theories of encapsulation focus on the manner in which processes are encapsulated as objects. However, the individual's perception of the original objects plays a vital role. Counting starts with objects perceived in the external world which have properties of their own: they may be round or square, red or green or both round and red. These properties need to be ignored if the counting process is to be encapsulated into a new entity—a *number* which is *named* and given a *symbol*. It is our contention that *different perceptions of these objects*, whether mental or physical, *are at the heart of different cognitive styles that lead to success and failure in elementary arithmetic*.

Three themes dominate the empirical studies used in building the resulting theory:

- differing cognitive styles reflected by children's approaches to elementary number combinations when they could not recall solutions.
- process/concept links as represented by the tactics used to carry out elementary computations.
- the nature of any imagery associated with these tactics.

## Differing Cognitive Styles

Gray (1991) built on the classification of children's solution strategies for solving addition and subtraction problems (Carpenter et al., 1981; Carpenter & Moser, 1982) which was, in turn built on the classification including count-all (CA), count-on (CO), derived fact (DF) and known fact (KF) (Groen & Parkman, 1972; Groen & Resnick, 1977). This research had two aims: to consider whether the classification in contextual situations could be transferred to context-free ones and to identify fall-back strategies chosen by children when they failed to know a fact—in short, to identify a cognitive hierarchy. Herscovics & Bergeron (1983) had emphasised that any such cognitive hierarchy need not apply to specific individual children. Gray found that the sequence of fall-back strategies revealed a divergence in thinking between different individuals.

The assumption was that if the child preferred to solve numerical problems by remembering the answer (known fact), if this is not known, the most efficient alternative is to use another known fact to derive a solution. Should both these strategies fail, it was assumed that the child will resort to the next preference by counting. Logic, supported by evidence from other work (e.g. Fuson, 1982; Secada et al., 1983; Steffe, et al., 1983) seemed to indicate that the descending order of preference, theoretically available to all children, could be considered as a direction of regression.

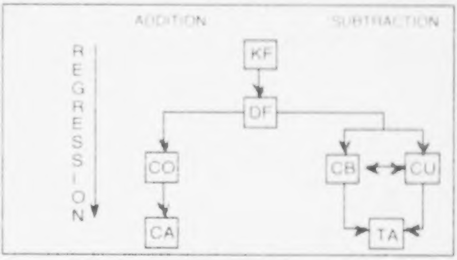

Figure 1: Theoretical Model of children's strategy regression when solving context free elementary number combinations

Figure 1 represents a model of this regression for addition and subtraction in which count-back (CB) and count-up (CU) are indicated as alternatives (Woods, Resnick & Groen, 1975). It is implicit in this range of strategies that the child's use of counting methods could reveal something more about understanding of counting than the mere use of a procedure to solve the problems. For example, children who use count-all or count-on as dominant strategies for addition often see subtraction as the inverse of these operations. Whilst such strategies are necessary pre-conditions for a child to relate addition and subtraction (Steffe et al., 1983), it was hypothesised that children using either display qualitatively different thinking from those only using one or the other.

The evidence is based on responses to a range of elementary context-free addition and subtraction problems given by 72 children from two schools. Identified by age ( 7+ to 11+) and achievement ('above average', 'average' and 'below average'), the children's responses demonstrate that the above classifications are suitable for context-free items. The results indicate that some children wish to remain at a procedural level which, in terms of information processing, make things very difficult for them, whilst others operate at a conceptual level which is more flexible. The notion of different cognitive styles leading to a diverging outcomes came from the observation that the less able, who relied extensively on counting procedures, were "*making things more difficult for themselves and as a consequence become less able*" (Gray, 1991, p. 570) whilst in contrast, the ability to "*compress the long sequences [of procedures] appeared to be almost intuitive to the above-average child*" (*ibid.*).

## Process Concept Links and the Proceptual Divide

Drawing upon the children's interpretations of symbolism, the differing cognitive styles evident in this first study were later placed within the context of a proceptual divide between those children who processed information in a flexible way and those who invoked the use of procedures. Those doing the former have a cognitive advantage. They link procedures to perform arithmetic operations with number concepts through cognitive links relating process and concept. Two pieces of evidence seem to support the notion that these differences are manifestations of qualitatively different thinking. The first considers the cumulative responses made by children in the above study to subtraction combinations for which they could not recall a solution. Figure 2 shows the response rate to various number combinations. (The 'known fact' responses are omitted, but implicit, since the responses total 100% in each case.) Though distinct age groups are not identified within this figure the general distinctions are clear to see. The left hand side shows how the high achievers use almost all derived facts and a few examples of counting, whilst the right hand side shows few derived facts and a large percentage of counting. The proceptual divide is clearly shown.

It may be argued that discrete "snapshots" of children fail to take account of the stage theory proposed by Piaget. This suggests that given time all children go on from pre-operational to concrete operational and finally to formal operational thinking. This theory implies that all children should be able to encapsulate counting procedures into numerical objects. Observation within any classroom shows that this is not the case.

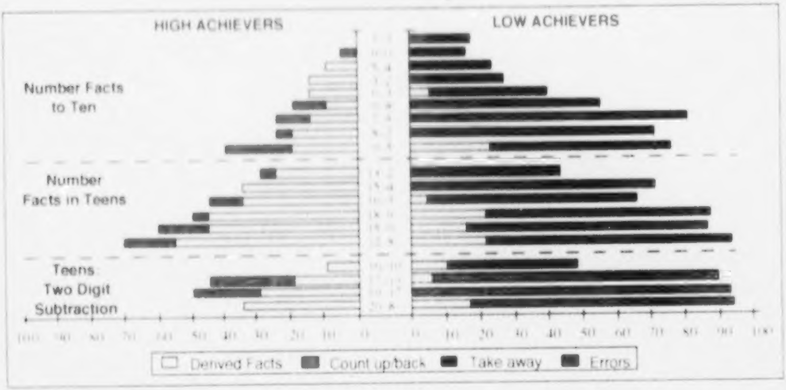

Figure 2: Strategies used by children of differing achievement to solve context-free subtraction combinations when the solution could not be recalled (adapted from Gray & Tall, 1991).

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The ability to simply recall facts may muddy the theoretical waters. Far more significant is the way in which the children may use the facts they already know to establish those as yet unknown. Recognition of this difference may allow us to distinguish between those facts that are isolated pieces of knowledge and those that are usefully connected to others. Trends pointing to longer term differences were confirmed by a small scale longitudinal study (Gray, 1993).

A group of 29 children (from a different school) were grouped according to their level of achievement in the numerical components of the Standard Assessment Tasks (SEAC, 1992) given to all children within the UK at the end of Key Stage 1 (7+). These tests identify levels of competence normally expected of the "average" seven year old and may also be used to identify children at each extreme of the spectrum of achievement. The children were interviewed individually after the test on a range of context-free elementary number combinations that formed the basis for the test. The same children were interviewed one year later on the same items and on items which reflected their mental approach to two digit addition and subtraction.

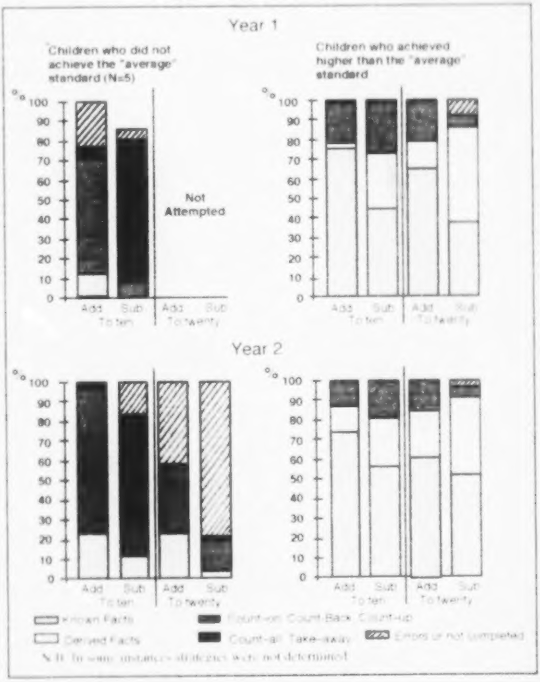


Figure 3: Solving elementary number combination: strategies used by "low" and "high achievers" over a two year period.

The results of the elementary components (figure 3) show how over the two interviews counting procedures, frequently very inefficient, dominated the strategies used by children who did not reach the average standard. In the second phase of interviews the children's counting approaches were sufficiently robust to cope with all combinations to ten but they remained unreliable for combinations to twenty. There is an extensive use of derived facts by those who achieved an above average level of achievement.

Although as teachers we often ask children what, to us, appears to be the same question, various children may interpret it very differently. The expression 4+3 actually signals children to do different things. To some it is a concept to know. To others it is a process to do. It is conjectured that such differences may be manifestations of different stages of cognitive development in various children:

> When a procedure is first being learned, one experiences it almost one step at a time; the overall pattern and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity–it becomes a thing. It, itself is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. (Davis, 1984, p 29–30)

However their 'permanency' may also a reflection of different cognitive styles reflected in, for example, the cognitive shift associated with encapsulating the process of addition as the concept of sum. Within figure 4 we see this as the result of the qualitative compression of the lengthy count-all procedure into the shorter one that is count-on. The evidence seems to suggest that different cognitive styles may lead to the bifurcation in thinking that is evident in the proceptual divide.

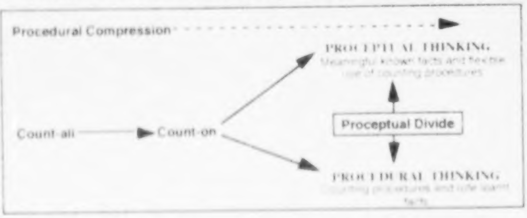

Procedural Compression
PROCEPTUAL THINKING
Meaningful known facts and flexible use of counting procedures
Proceptual Divide
Count-all
Count-on
PROCEDURAL THINKING
Counting procedures and rote learnt facts

Figure 4: Procedural compression associated with numerical development and the formation of a proceptual divide for early number.

The common pedagogical approach to numerical processes builds on the belief that number development should commence with enactive approaches and that, given sufficient time, all children will encapsulate arithmetical processes into numerical concepts. The existence of a proceptual divide would seem to indicate that this is *not* the case and even when teaching programmes have been designed to shift the lower achievers focus from processes to thinking strategies (see, for example, Thornton, 1978) lower achievers resist a change from the security offered by their well known counting procedures. Further, we conjecture that positive efforts to make the relationships implicit in proceptual thinking explicit to those that do not have the associated flexibility run the danger of being seen by some as a new set of procedural rules.

So what causes the proceptual divide? We may conjecture that pedagogy may account for it in some degree. There does exist a certain 'conspiracy' between pedagogue and learner which is manifest in the belief that being shown how to do something solves current difficulties (see, for example, Skemp, 1977). We conjecture that one cause of the proceptual divide is the qualitatively different focus of attention which, on the one hand places the emphasis upon concrete objects and actions upon these objects, and on the other on abstraction and the flexibility intrinsic within the encapsulated object. Why is it that some children seem to implicitly recognise this power but others do not?

### Imagery and Elementary Arithmetic

To gain a partial answer to this question our attention turned to imagery. Our fundamental thesis was that different qualities of mathematical abstraction were influenced by the child's cognitive style and that the relationship between achievement and qualitative difference may be determined by considering:

- the nature of the object that was dominant in children's imagery
- the way imagery is used within elementary arithmetic.

Psychological research has identified the importance of imagery in cognitive development and children use it more in their thinking than adults (Kosslyn, 1980). Its role in the child's thought processes, cause it to have far-reaching consequences on children's concepts and reasoning (Bruner, Oliver & Greenfield, 1966; Piaget & Inhelder, 1971) and therefore images place major constraints on cognitive processes.

The relationship between different forms of representation may be seen through the presentation and solution of arithmetic facts (Deahenne & Cohen, 1994) and in the context of arithmetic mental representations of the objects will effect mental operations. (Gonzalez and Kolers, 1982). Children's internal representation of numbers are often highly imaginative and unconventional and built up over time (Thomas, Mulligan & Goldin, 1995) but the possession of an image of a mathematical idea implies that the individual does not need actions or the specific instances of image making (Pirie & Kieran, 1994). However, they may be eidetic in the sense they can be visual representations of previously scanned material (Leask, Haber & Haber, 1969) and fully formed from something presented (Mason, 1992) though classification of this phenomena is a problem (Gregg, 1990).

To associate the notions of achievement and 'qualitative difference' with the role of imagery we make the assumption that an image is mediated by a description (Kosslyn, 1980; Pylyshyn, 1973). Following Pylyshyn we make the assumption that the representation conforming to an image is more like a description than a picture. The classical notion is usually of a visual image—though images can be formed from other modalities—which appears to have all of the attributes of actual objects or icons. In the context of numerical development Seron et al. (1992) suggest that images of quantities directly represented by "patterns of dots or other things such as the alignment of apples or a bar of chocolate" (p. 168) may be deemed as analogical.

## Methodology

Paivio (1991) suggested that the generation of an image promotes the development of a trace in the brain that integrates the separate components of the item in question. Accessing a part of the information encoded in memory prompts the retrieval of all other pieces of information contained in the image (Woloshyn, Wood & Pressley, 1990). To gain a sense of the nature of children's imagery associated with both concrete and abstract objects and the relationship this may have with mentally processing elementary number combinations, 24 children, selected to represent the extremes of ability, 'low achievers' and 'high achievers', across four age groups, 8+ to 12+, were first asked to respond to auditory and visual items and then asked to provide mental solutions to a series of elementary arithmetic combination in addition and subtraction.

The research methodology used semi-structured clinical interviews (see also Gray & Pitta, 1996; Pitta & Gray, 1996). Items which prompted discussion were presented in a way that gave the interviewees the freedom to follow their own inclinations. Data from each individual was collected in a variety of ways including records of achievement and teacher assessment. The initial selection of children was made from full class records. Each individual interview was audio and video taped and subsequent transcriptions formed the basis for response classification. When responding to each item within the auditory and visual sections, children were asked to provide a first notion of 'what came to mind' when they first heard or saw the item. They were then given 30 seconds to talk aloud about the item in question. For auditory items, children were also asked to explain the item so that an extra-terrestrial may understand what it was.

The items within the auditory section contained words such as 'ball', 'car', 'triangle', 'five', 'fraction' and 'number'; in the visual section were icons representing 'two quarters', a 'dancing man', geometric shapes forming a 'house', and so on, and symbols such as '5', and '3÷4'.

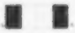
### Qualitative differences in interpretation

Though there are a wide variety of conclusions that may be drawn from each item, the analysis of the results indicates that similarities in the children's descriptions of imagery are remarkable both for their consistency across the range of items, and for the differences they displayed between the 'high achievers' and the 'low achievers'.

When responding to the auditory items, the 'low achievers' tended to highlight the descriptive qualities of items in strongly personalised terms, qualities also evident when the children responded to the visual items. However, there was a tendency to associate these items with a story in the sense that they were seen as pictures that required colour, detail and a realistic content. In contrast, 'high achievers' concentrated on the more abstract qualities within both series of items. Though they initially focused on core concepts, they could traverse at will a hierarchical network of knowledge from which they abstracted these notions or representational features. An overall summary of the analysis of the children's responses to the auditory and visual items is given in table 1. Each item triggered 'low achievers' to provide descriptions which were qualitatively similar whereas 'high achievers' used each comment to trigger a qualitatively different comment. For example, amongst the 'low achievers' the notion of 'ball' consistently evoked images from boys of 'scoring a goal' or playing in the school team whilst

| | Low Achievers | High Achievers |
|---|---|---|
| WORDS | • Concretised<br>• Unable to reject information<br>• Horizontal thinking directed towards surface features<br>• *Imitation* | • Focus on abstract qualities<br>• Able to reject information<br>• Vertical thinking directed towards core features or definitions<br>• *Thought generator* |
| ICONS | • Interpreted as a 'picture' out of focus, an incomplete concrete reality requiring focus<br>• Given colour, detail and realism (with imagination)<br>• Display 'horizontal' thinking– imaginary extensions similar in quality<br>• *Imitation* | • Concentrate on abstract qualities<br>• Ignore detail–concentrate on interpretation<br>• 'Vertical' thinking – free movement between abstract and descriptive aspects<br>• *Thought generator* |

Table 1: A comparison of children's interpretations of words and icons.

the visual item 'two quarters' , was frequently identified as curtains on a window and the detail was added by describing the curtains as "flowery", "rose coloured" or "green".

Responses to symbolic items (table 2) bore striking similarities to those with words or icons and reflected the degree within which the children were involved with the abstract qualities of the objects. The higher the involvement, the more the child was able to talk about the items at an impersonal level. On hearing the word 'five' or 'half', 'high achievers' often referred to the symbol using phrases such as "it is" to illustrate semantic aspects of the object. For example, the word 'five' drew responses such as "it is two plus three, one hundred take away ninety five", or "it is prime because it is only divisible by one and five". This does not mean to say that they did not attach qualities arising from episodic memory, such as "I had five candles on my cake for my fifth birthday"; high achievers' were able to do both. On the other hand 'low achievers' almost always displayed examples of episodic memory, concretised the item, "I have five fingers", or associated its use with some arithmetical action such as counting.

| Low Achievers | High Achievers |
|---|---|
| • Order to carry out an action | • Recognised as both the holder of an idea and an action |
| • Concretised by either<br>(a) associating with a concrete item or<br>(b) identifying as an icon | • Detached from concrete qualities, associated with abstraction |
| • 'Horizontal thinking' demonstrated through procedural association | • 'Vertical thinking' demonstrated by proceptual flexibility |
| • *Imitation* | • *Thought generator* |

Table 2: A comparison of children's interpretation of symbols

Comparing table 2 with table 1 indicates how imagery associated with non-arithmetical objects may carry similarities with imagery of named arithmetical objects and symbols. Such similarities may be summed up by concluding that images of the low achievers are episodic and active, whilst those of the high achievers are semantic, and generative. We use the terms 'episodic' and 'semantic' to draw a distinction between images arising from memory associated with the recollection of personal happenings and events and images associated with organised knowledge having meaning and relationships. The former is based upon access to previous experience, the latter no longer depend on learning episodes that provided the basis for knowledge (see Tulving, 1985).

The qualitatively different responses to the words, icons and symbols indicates that the 'low achievers' were reluctant to reject information and, if there was little to describe, they created it by establishing stories around the items using images from their known physical world, often as participants in the image, elaborating the detail whenever it seemed that such embellishment was required. In some instances they drew upon one image which acted as a symbol, for example, "my football", "my mother's car". The objects referred to were invariably real, quantifiably different, but qualitatively the same. In contrast, 'high achievers' filtered out the superficial to concentrate on the more abstract qualities of the items. Though they focused on real world concepts, they were also able to relate to a hierarchy of ideas which allowed them to refer to objects in the abstract by using qualitatively different notions or representational features.

### Images in Elementary Arithmetic

Such differences became marked when images associated with children's responses to the range of elementary number problems were considered. Again 'low achievers' tended to concretise and focus on all of the information. Symbols were translated into numerical processes supported by the use of figural objects that possess shape and in many instances colour. Transformation of numerical symbols into mental analogues of physical objects took two general forms. The first was as 'dots' or 'marbles' (see figure 5, adapted from Pitta & Gray, 1997). The use of such objects frequently involved mental processes akin to subitising. As may be seen from the diagrammatic representations arrays of dots supported the mental activity but in all instances where they were described such images were limited and there was no evidence of their use on numbers in excess of ten.

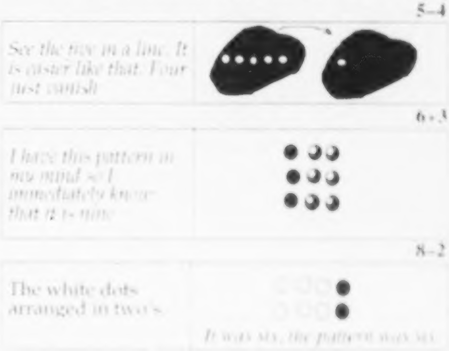

5–4
*See the five in a line. It is easier like that. Four just vanish*
6+3
*I have this pattern in my mind so I immediately know that it is nine*
8–2
The white dots arranged in two's
*It was six, the pattern was six*

Figure 5: Analogical images based upon discrete objects

Frequently 'low achievers' reported imagery strongly associated with the notion of number track although the common object which formed the basis of each 'unit' of the track was derived from fingers. In some instances children report seeing full pictures images of fingers, in others it was 'finger like'. The essential thing is that the object of thought was 'finger' and these invoked a double counting procedure. Other images used to support double counting were dynamic invoking actions. Figures 6 and 7 indicates diagrammatic copies of representation given by a nine-year-old and an eleven-year-old. These are associated with the solutions to 9–5 and 7+4 .

In the first we see the dynamic image that grows from a pattern of nine. The procedure used was count-back and as each counter was counted it was moved and assigned a new numerical value. When the count back of five had been completed the child knew from the pattern "that 3 and one makes four".

Within figure 7 we see how each phase of the solution procedure evolved from the previous one. First the "black" seven appeared with "four white balls". One of the balls had an eight written above it and the

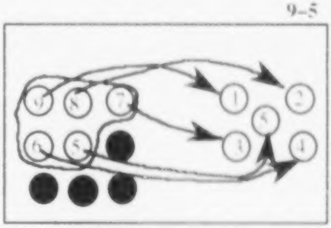


Figure 6

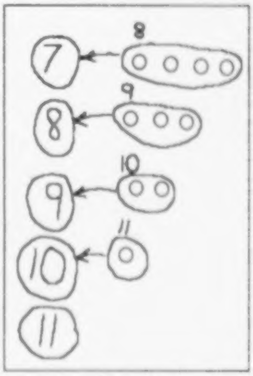


Figure 7

eight moved to take the place of the seven which disappeared. There were now three white balls the one nearest the eight having a nine written over it. This now moved to take the place of the eight, and so on.

Such images were essential to the action; they maintained the focus of attention. The objects of thought of the 'low achievers' were analogues of perceptual items that seemed to force them to carry out procedures in the mind, as if they were carrying out the procedures with perceptual items on the desk in front of them. When the image failed they used the real items. For these children mathematics involved action and to carry out the action they used real things.

Symbolism enables us to utilise short term memory to better effect but the differences between the 'low achievers' imagery associated with symbolism and that described by the 'high achievers', was stark. It is here that we may see clearly the 'low achiever's' inability to filter out information thus providing the contrast between their uneconomical use of memory and the 'high achievers' economic use. Here, we should explain that we use the word 'economic' not simply to illustrate differences in the detail but also in arrangement as well as quality.

Symbolic images played considerably less part in processing for 'low achievers' than they did for 'high achievers'. It was also reported far less than analogical images.

Figure 8 shows an eight-year-old child's diagrammatic representation of imagery associated with 3+6. The child described all the numbers going around in his head in circles. "The number I want moves out and I count them. Then they go back and new numbers go out." In this case it was first the '3' and the '6'. These became "blacker" than the other numbers. The three moved back and became four and the six moved back and became 7. For this child such imagery was only associated with number combinations to ten. For the other numerical items perceptual units were used.

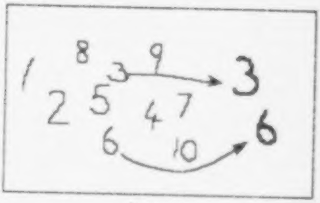


Figure 8

The notion of "spinning" seemed to be a common feature of the 'low achievers' descriptions, implying that images remained for some time and possessed movement. Even when adding 2+1 a nine year old reported seeing all of the operation symbols "spinning around on one side and a big black 3 on the other". In some instances images were associated with approximation. When adding 6+3 another nine-year-old reported seeing "a jumble of numbers with 8 and 9 standing out because they are near the answer." This was a similar response to that given by a twelve-year-old who, when doing the same combination reported an image that consisted of 3,6,9,12,15, and 18. "All the numbers were in the three times table". Whilst the "three and the six stayed there because they were part of the nine, the twelve, fifteen and the eighteen just fall away."

The use of symbolic imagery amongst 'high achievers' was far more economical. The word "flashing" dominated their descriptions instead of "spinning". Images came and went very

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quickly. "I saw '3+4' flash through my mind and I told you the answer", "I saw a flash of answer and told you." It was not unusual for the children to note that they saw both question and answer "in a flash", sometimes the numerical symbol denoting the answer "rising out of" the symbols representing the question. In instances where children reported the use of derived facts it was frequently the numerical transformation that 'flashed'. For instance when given 9 + 7 one eleven year old produced the answer 16 accompanied by the statement. "10 and 6 flashed through my mind."

## Discussion

Clearly the quality of imagery generated differs considerably. On the one hand we see the dominant objects being either physical, such as fingers and counters, or figural representations of physical items. On the other we see it as an object of thought. 'Low achievers' concentrate on analogues of physical actions, and where they use symbolism they continue to carry out actions associated with such analogues. Their images are not so much associated with "knowing" mathematics but with "doing" mathematics. In contrast the symbolic images of 'high achievers', appear to act as though generators. They appear to flash as memory reminder's, momentarily coming to the fore so that new actions or transformations may take place.

| Low Achievers | High Achievers |
|---|---|
| •Concretised<br>Unable to reject information | •Focus on abstract qualities<br>Able to reject information |
| • "Horizontal" thinking directed towards procedural associations through variations of the figural / imaginary items | • "Vertical" thinking. Attention directed towards known facts or transformations |
| •Imitation | •Thought generator |
| •*Excessive memory, overload of working memory* | •*Economic use of working memory through use of symbols* |

Table 3: Children's imagery in an arithmetical context

The evidence that comes from children's imagery associated with elementary arithmetic combinations is given in table 3. Comparison with table 1 and 2 clearly shows the similarities and differences between the two groups of children over the range of items that formed the basis for comparison. We once again see the tendency of 'low achievers' to concretise and focus on all of the information. Imagery in the numerical context is strongly associated with procedural aspects of numerical processes. The children carry out procedures in the mind as if they were carrying out procedures with perceptual items on the desk in front of them. 'High achievers' appear to focus on those abstractions that enable them to make choices. Their ability to reject information is again apparent. We suggest that such differences have overriding consequences for children's mathematical achievement. The one conclusion that may be drawn for the use of analogical images is that it would seem to place a tremendous strain on working memory. Gear et al. (1991) have suggested that a component of developmental difficulties in mathematics is a working memory deficit. We would suggest that on the contrary these low achievers show an extraordinary use of working memory. Their problem is one associated with its use and not its capacity. Not only is the child focusing on the representation but also on discrete numbers in that representation.

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The ability to filter out information and see the strength of such a simple device as a mathematical symbol appears to be confined to the high achievers. The evidence suggests that children who are 'low achievers' in mathematics appear unable to detach themselves from the search for substance and meaning—no information is rejected, no surface feature filtered out.

We believe that this has serious consequences which contribute to the formation of the proceptual divide. The notion of procedural compression and the interiorisation of mathematical processes is strongly embedded in the literature. Interpretations of Piagetian notions that enactive approaches will form a foundation for procedural encapsulation may be associated with Bruner's (1968) view that past experience may be conserved through such enactive approaches. Of course, whilst the latter must also be seen within the context of iconic and symbolic conservation, it would seem that far from 'encapsulating' enactive interpretations of arithmetical processes, the 'low achievers' are mentally imitating them.

The quality of image formed from enactive approaches is dependant upon what it is the child chooses to create an image of. This will influence the use to which the image is put. It is conjectured that this will not only have consequences for the quality of the action that is taken into consideration but it will also affect the quality of the object which dominates the child's imagery. It would seem reasonable that if some children concentrate on actions with physical objects and work hard to develop competence with these actions the more they are likely to use them.

Such considerations add a new quality to the notion of proceptual divide, one that is so strongly associated with image formation that it is possible that children's interpretations of mathematical actions may be strongly influences by their interpretations of their real world. In early mathematics children are faced with not one but *two* interpretations of their interaction with externally perceived objects. On the one hand it is the identification of the qualities of objects that arise from manipulation and perception which lead eventually to the development of geometrical concepts. On the other, though perception and manipulation are the dominant actions, it is the cognitive shift associated with the result of these actions that brings about the development of numerical concepts. The objects that are the catalysts for both strands of development are the same but the conceptual development is different. We believe that this has serious implications for pedagogy. Early years within school are dominated by enactive methods in the belief that given the appropriate experience all children will "encapsulate" arithmetical processes to form arithmetical concepts. *Observation within any classroom shows that this is not the case.* Children may be focusing on *different* aspects of their experience. For some the dominant focus is on objects and the actions *on those objects*, others are able to focus more flexibly on the results of those actions expressed as number concepts. The former may seek the security of counting procedures on objects rather than the longer-term development of flexible arithmetic. We need to determine which, so that we may provide the necessary support both to those who develop flexibly and also to those who, at the very start of their mathematical development, appear to be travelling along the wrong road.

## REFERENCES

Beth, E. W. & Piaget, J., (1966). *Mathematical Epistemology and Psychology*. (W. Mays, trans.). Dordrecht: Reidel.

Bruner, J. S., (1968). *Toward a theory of instruction*. New York: W. W. Norton & Co.

Bruner, J. S., Oliver, R. O., & Greenfield, P. M., (1966). *Studies in cognitive growth*. New York: Wiley.

Carpenter, T. P., Hiebert, J. & Moser, J. M., (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education* 12, 27-39.

Carpenter, T. P. & Moser, J. M., (1982). The development of addition and subtraction problem solving skills. In T. Carpenter et al. (Eds.) *Addition and Subtraction: A Cognitive Perspective*, (pp. 9-24). Laurence Erlbaum: New Jersey.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: beginning with a coordinated process schema. *Journal of Mathematical Behavior*, 15, 167–192.

Davis, R. B. (1984). *Learning mathematics: the cognitive science approach to mathematics education*. Norwood, NJ: Ablex.

Dehaene, S. & Cohen, L., (1994). Towards an Anatomical and Functional Model of Number Processing. *Mathematical Cognition*, 1, 1, 83–120.

Dienes, Z. P. (1960). *Building up mathematics*. Hutchinson Educational: London.

Dubinsky, E., (1991). Reflective Abstraction. In D. O. Tall (Ed), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers. (1991).

Fuson, K. C., (1982). An analysis of the counting-on solution procedure in addition. In T. Carpenter et al. (Eds.), *Addition and Subtraction: A Cognitive Perspective*. Laurence Erlbaum: New Jersey.

González, E.G. & Kolers, P.A., (1982). Mental manipulation of arithmetic symbols. *Journal of Experimental Psychology: Learning, memory and cognition*, 8, 308-319.

Gray, E. M., (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational studies in mathematics*, 22: 551–574.

Gray, E. M & Tall, D. O., (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 25, 2, 115–141, (1994).

Gray, E. M. & Pitta, D., (1996). Number Processing: Qualitative differences in thinking and the role of imagery. In L. Puig and A Gutiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol. 4, pp 155–162. Valencia: Spain.

Gregg, V., (1990). Eidetic Imagery. In M. W. Eysenck, (Ed.), *The Blackwell Dictionary of Cognitive Psychology*. Blackwell: Oxford.

Groen, G. J. & Parkman, J. M., (1972). A chronometric analysis of simple additio'. *Psychological Review*, 79, 329-343.

Groen, G. J. & Resnick, L. B., (1977). Can preschool children invent addition algorithms?, *Journal of Educational Psychology*, 69, 645-652.

Herscovics, N. & Bergeron, J. C., (1983). Models of Understanding. *Zentralblatt für Didaktik der Mathematik*, 83, 75-83.

Kamii, C., (1985). *Young children reinvent arithmetic*. New York: Teachers College Press.

Kosslyn, S. M., (1980). *Image and mind*. Cambridge, Mass: Harvard University Press.

Leask, J., Haber, R. N., & Haber, R. B. (1969). Eidetic imagery in children: II Longitudinal and experimental results. *Psychonomic Monograph Supplements*, 3, 25–48.

Mason, J. (1992). Doing and construing mathematics in screen space. In B. Southwell, B. Perry and K. Owens (Eds.), *Proceedings of the 15th Annual Conference of the Mathematics Education Research Group of Australasia*, 1–17. University of Western Sydney: Nepean, Australia

Paivio, A. (1991). Dual coding theory: retrospect and current status. *Canadian Journal of Psychology, 45*, 255–287.

Piaget, J. & Inhelder, B., (1971). *Mental imagery in the child*. New York: Basic.

Piaget, J., (1965). The *child's conception of number*, (C. Gattengo & F. M. Hodgson, trans.). London: Routledge & Kegan Paul.

Piaget, J., (1971). *Genetic Epistemology*, (E. Duckworth, trans). New York: Columbia University Press.

Piaget, J., (1973). Comments on mathematical education. In A. G. Howson (Ed.), *Developments in Mathematical Education: Proceedings on the Second International Conference on Mathematics Education*. Cambridge, Cambridge University Press.

Piaget, J., (1985). *The Equilibrium of Cognitive Structures*. Cambridge MA: Harvard University Press.

Pirie, S., & Kieran, T., (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational studies in mathematics*, 26, 165–190.

Pitta, D. & Gray, E. M., (1997). In the Mind ... What can imagery tell us about success and failure in arithmetic? In G. A. Makrides (Ed.), *Proceedings of the First Mediterranean Conference on Mathematics*, pp 29–41. Nicosia: Cyprus.

Pitta, D. & Gray, E. M., (1996). Nouns, Adjectives and Images in Elementary Mathematics. In L. Puig and A Guttierrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol. 3, pp 35–42. Valencia: Spain.

Pylyshyn, Z. W., (1973). What the mind's eye tells the mind's brain. *Psychological Bulletin*, 80, 1–24.

Schools Examination and Assessment Council, (1992). *Standard Assessment Task, Pupil Sheet Booklet, Key Stage 1*. Lincoln: NFER-Nelson.

Secada, G. W., Fuson, C. K. & Hall, J. W., (1983). The transition from counting-all to counting-on in addition, *Journal for Research in Mathematics Education*, 14, 47-57.

Seron, X., Pesenti, M., Noel, M-P., Deloche, G., & Cornet, J-A., (1992). Images of numbers, or "When 98 is upper left and 6 sky blue". *Cognition*, 44, 159-196.

Sfard, A., (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Skemp, R. S., (1976). Relational understanding and instrumental understanding, *Mathematics Teaching*, 77, 20-26.

Steffe, L., von Glaserfeld, E., Richards, J., Cobb, P., (1983). *Children's Counting Types: Philosophy, Theory and Applications*. Preagar: New York.

Tall, D. O. & Vinner, S., (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

Thomas, N., Mulligan, J., & Goldin, G. A., (1996). Children's representation of the counting sequence 1–100: Cognitive structural development. In L. Puig and A Guttiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol. 4, pp 155-162. Valencia: Spain.

Thornton, C. A., (1978). Emphasizing thinking strategies in basic fact instruction. *Journal for Research in Mathematics Education*, 9, 214-227.

Tulving, E., (1985). How many memory systems are there? *American Psychologist*, 40, 385–398.

Woloshyn, V., Willoughby, T., Wood, E. & Pressley, M. (1990). Elaborative interrogation and representational imagery facilitate adult learning of facts presented in paragraphs. *Educational Psychology*, 82, 513–524.

Woods, S. S., Resnick, L. B. & Groen, G. J., (1975). An experimental test of five process models for subtraction. *Journal of Educational Psychology*, 67, 17-21.

# POSTULATED COGNITIVE PROCESSES IN MATHEMATICS

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When I first began to study Piaget, I was struck by the powerful way that his theory could be turned around and applied to itself. Those of us studying Piaget's works came to the task with our own assimilation paradigms, and were confronted by the same challenges of assimilation and accommodation that Piaget was postulating in his effort to understand how children built up their personal knowledge of space, time, and quantity. This same kind of reflexivity applies also to our present study of the paper by Gray, Pitta, and Tall, and to the important analysis by Dagmar Neuman. We come to this task with different personal repertoires of knowledge, and it is inevitable that we see these two papers in our own personal ways.

I read these papers as dealing with two fundamental challenges facing the psychology of mathematics learning today: first, how ought we to interpret the commonly-observed fact that some students seem to find the study of mathematics nearly effortless, whereas other students seem to find it virtually impossible? Second, what kind of underlying theoretical structure can we postulate that may help us build up our understanding of mathematical thinking? It seems to me that the two papers approach this task from two very different but equally valuable points of view: Gray, Pitta, and Tall wish to answer the first question, while keeping much of their attention focused on building up a postulated theory. Neuman appears to approach the same first question, but with her attention mainly on very specific details of the information processing tasks that children face in early arithmetic thinking. It is rather as if Gray, Pitta, and Tall are working to create the Periodic Table of chemistry, while Neuman wishes to gain a detailed knowledge of the chemical behavior of hydrogen. Both are equally worthy, support one another, and are not inherently in conflict - but they do confront readers with different kinds of challenges.

To what extent do all of us in the Research Forum share a common set of underlying assumptions? Perhaps we agree that:

- every person has his or her own individual repertoire of assimilation paradigms (if I were to say the word "dog" every one of us would have different ideas in response to this word - in my case, I would think of a standard black poodle named Eudoxus, and I would think that I had loved him very much, but I am probably the only one here for whom the word "dog" would bring forth exactly these thoughts);

- the items in this repertoire have been built up from past personal experience;
- when confronted by some sort of task or other demand, every person has at least some potential flexibility or possibly-available alternatives in choosing among different assimilation paradigms (whether we do this consciously or not); if you ask me to add 28+4, I might write it down and use a standard addition algorithm, I might visualize a number line and "see" how to move 4 units to the right of 28, I might decompose the problem into sub-problems, etc.;
- in studying mathematical behavior, it is valuable to try to analyze the processes that a student might use in order to perform some specific task;
- there appear to be pronounced differences in student behavior, particularly in relation to some kind of progression from concrete ways of thinking, toward more abstract ways of thinking;
- there appear to be pronounced differences in the **kinds** of approaches that different students take to tasks;
- and, in light of the last two items, it may make sense to postulate different kinds of entities that different people use as their mathematical thinking develops from experience or from instruction (no matter which of the various terms you choose from the list presented in Gray, Pitta, and Tall).

Like the axioms for the integers, these assertions may sound innocuous - but, like the axioms for the integers, they have important implications.

- They would seem to imply that we should not limit our search for mental processes to seeking "the one basic pattern that everyone uses", but - on the contrary - we should expect that the processes used by a student may well have been influenced by past experiences of that individual student.
- They also seem to imply that whenever we believe we are witnessing unsatisfactory thought processes in students, we need to ask ourselves how we might change our instructional programs to remediate (or, better, to avoid) these less-than-optimal ways of thinking.

It may be advisable to select some neutral terminology, carrying the least possible baggage of prior connotations. For all of the assimilation paradigms, all of the known facts, all of the established algorithms that a person's mind may contain, we need a neutral term. For the moment I propose to call any of these a "knowledge item". One way, then, that people differ is that each of us has a different repertoire of knowledge items. (Maybe this sounds more meaningful if we state it as: "every one of us has our own distinct repertoire of assimilation paradigms" - like Eudoxus the poodle, in my case, or the idea that $4 - 8 = {}^{-}4$, which Kye, a third grade student in Weston, Connecticut used in inventing an original algorithm for $64 - 28 = ?$ [Cochran, Barson, and Davis, 1970].)

But there is another important distinction: when a specific person is confronted with a specific situation, there are different possible ways in which the individual may interpret what needs to be done. Confronted, say, with the task of "learning" the definition of continuity of the function $f(x)$ at the point $x=x_o$, some students on some occasions will try to memorize the words, while largely ignoring the meaning; other students (or the same students at other times) begin immediately to try to get a grasp on the meaning, perhaps by using graphical representation, or by trying to re-state the definition in their own words, etc. (In one extreme case, a university student, told that he "needed to study logarithms," spent several days trying to memorize the numbers in a table of logs.)

The observed differences between students who are mathematically able, vs. students who are mathematically less able, have traditionally been explained, in one way or another, in terms of some forms of innate ability. But suppose, instead, that the differences are in substantial part attributable to differences in individual repertoires of knowledge items, or in differences in how a task is perceived. We are not speaking here of the obvious difference that some people know more than other people know, but rather we are referring to deeper differences in the kinds of things people may know, differences that may lift one person's thinking to different levels of strategy or different kinds of cognitive activity.

The two papers introduce an intriguing hypothesis: because the differences between students are evident at an early age, they may be influenced by pre-school learning. But this pre-school learning is probably informal, and may take place where the child is attending to the task itself, and not to the expectations of a teacher. The child might, say, really want to solve some quantitative problem, whether with counters or tea cups or pieces of candy, or whatever. The child's perception is centered on the tea cups or the candy, and the child's actions are directed toward an effective solution of the problem. This may be an ideal situation in which mental imagery and appropriate actions (and perhaps also discussion with others) can become welded together into a valuable knowledge item that is thereafter available in the child's mental repertoire.

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This could contrast strikingly with a classroom setting where a child is attempting to meet a teacher's expectations, and where the emotional stress may be far greater, and the child's attention may be focused on the teacher, not on the task or on possibly-useful ideas. The informal settings of pre-school learning may enable some children to form a far more useful collection of mathematical ideas.

If so, what can schools hope to do? Citing Neuman, 1987, 1994, and Yackel & Cobb, 1996, Neuman points out important possibilities:

> That those ideas are formed outside school does not mean that it is impossible to organize formal teaching in a way where they can be invented. There are many examples of mathematics classrooms where children structure objects in contexts of meaningful play and games, and where they ... [interact with others, and with materials] and argue for what they do, getting the opportunity together to explore the structures they have formed...

So: One thing we have in the papers of Gray et al., and of Neuman, is a theory that suggests some important differences in school instruction. It is interesting -- and encouraging -- that the changes suggested by these two papers are strikingly similar to the recommendations that are now becoming familiar in the United States through the NCTM publication usually called the Standards (NCTM, 1989). We have here part of a theoretical foundation supporting the Standards.

But of course our interest goes beyond that. We want to consider the theory itself. And what, exactly, does this theory explain? For one thing, it focuses on cognitive **transitions** - how, and when, children switch from one mode of thinking to another, as when children switch from "counting all" to "counting on" in dealing with addition (Fuson, 1982), or when children switch from one way of calculating 2+7=? to the more efficient use of 7+2=? (Groen & Resnick, 1977). There are many different ways we can try to conceptualize the processes that underlie these transitions:

- a familiar action, or sequence of actions, may come to be seen as a "thing" (Davis, 1984; Dubinsky, 1991; Sfard, 1991);

- actual physical objects (such as pebbles or poker chips) may come to be replaced by featureless "abstract" objects (so that we can visualize "two" of something without a need to see any specific things);

- a physical representation (for example, the 5-cm.-long yellow Cuisenaire rod) may be taken to stand for some implied action on, or arrangement of, other objects (in this case, 5 one-cm.- long white rods). Children do this, for example, when they are trying to find how many white rods would be required to construct a long train of specified length, they run out of white rods, and they make use of some yellow rods (which they count as "5"), or when they use the "undivided five" reported by Hatano (1982);

- number words, originally understood as "operators" (so that a child is happy to show you "three pencils", but seems to have no understanding of "three" by itself) come to be accepted also as numbers - that is to say, as abstract entities in their own right. This transition was the basis for Max Beberman's work with "shrinkers" and "stretchers" in the 1960's. As in many other transitions, what is involved here probably includes a sizable number of other activities and ideas - for example, the experience of marking numbers on a number line (Alston et al., 1994).

The third of these processes (accepting a yellow rod as representing 5 white rods) is, itself, presumably composed of several parts, since it involves accepting one single thing as representative of some potential action, and it also involves changing the focus of one's attention to some alternative **attribute** (in this case, becoming concerned with the new attribute of length, and seeing counting as a way of dealing with length, so that counting becomes a means to some other increasingly recognizable end - up until now, "length" may have been largely implicit and unexamined). One sees this transition, for example, when a child accepts a white rod as "one fifth of a yellow rod", but refuses to accept the 2-cm.-long red rod as "two fifths", on the grounds that the red rod "isn't two of anything - it's **one** thing" -- but then, after a moment of reflection, decides that calling the red rod "two fifths" is really acceptable after all (Alston et al., 1994).

All of these processes involve some form of "clumping" or aggregation. This process has to be a fundamental part of human cognitive growth, in no way confined merely to mathematics, as George Miller indicated in his paper "The Magic Number 7 ± 2" (Miller, 1956), where he pointed out that, if human thought is limited to about 7 simultaneous ideas, this can become less limiting if each "idea" can be made to consist of several component parts, a process that can presumably be used recursively.

It is important to appreciate the large aggregate of experiences and component ideas that may be grouped together to form some single new concept (see, for example, Davis & Vinner, 1986). Great effort often seems to be required to create these aggregations, but once they are securely established, it can be hard to imagine why they were not immediately obvious from the very beginning. (One mathematician claims to recall how, as a student, he had difficulty imagining "a sphere with one point removed" -- how was it different from a sphere? But of course as one comes to be familiar with closed and open sets as domains of functions, and other matters of that sort, the distinction between "a sphere with one point removed" vs. "a sphere" comes to seem entirely sensible and unproblematic.)

In order to build an appropriate postulated theory to help us understand these transitions to new forms of thinking, it would seem that we need to postulate some entities that are created within the human mind. (I am unclear as to why Gray et al. seem to be reluctant to take this step -- as, for instance, when they say: "In this sense there is no claim that there is a "thing" called a "mental object" in the mind." The postulation of not-yet-observed entities is an entirely familiar part of science, as in the case of "genes" in biology, or "electrons" and "atoms" in physics. I have always particularly treasured the postulation of "neutrinos," which were said to possess an impossible combination of attributes, but were none-the-less postulated to exist, the task of reconciling contradictions being wisely left as something to be worked out at some time in the future.) Surely, however we choose to describe it, there can be little doubt that, one way or another, a very large collection of knowledge items must often come to be welded or chunked or somehow bound together, after which they can be thought of as a single "new" idea.

But are these new aggregates some different kind of thing, of a sort quite unlike the component parts from which they have been constructed? Presumably not - the constituent pieces had been themselves constructed by aggregating earlier pieces, and these earlier pieces had themselves been built up as aggregates, and so on, back to some very early primitives that we cannot presently identify. (It is probably not helpful here to invoke Russell's Paradox, nor Russell's Theory of Types. The temporal sequential order, within the life of a single person, eliminates problems such as Russell's Paradox.)

Finally, there is the matter of how one sees one's task as a learner. In a recent teaching experiment at Rutgers, we have found adult learners who insist on dealing with many separate cases of simultaneous linear equations in an introductory algebra course: the case where you add two equations together to eliminate $x$; the case where you add two equations together to eliminate $y$; the case where you subtract one equation from the other, to eliminate $x$; the case where you subtract one equation form the other to eliminate $y$; the case where you multiply one equation and then add; and so on, for a very long list of "alternative methods". These particular adult students resist the idea that a more heuristic approach may be easier: (1) try to identify (as precisely as you can) exactly what is making the task difficult [in this case, either equation alone would have two unknowns, and we would have no way of dealing with this]; (2) perhaps using your answer to item 1, decide what you want to do (eliminate one variable); (3) plan how you may be able to do this; and so on. We have suggested that such "strategies" or "habits" or "predilections" or "interpretations of what one ought to be doing" (or whatever) - such as "trying to memorize a procedure as a step-by-step sequence" vs. "trying to understand an heuristic analysis" - may be in some way different from other kinds of "knowledge items" that we can retrieve from memory. But how sure are we that this is true? Perhaps a student can use this sort of heuristic strategic planning only if the student already has some assimilation paradigm to guide this planning. But then the matter of "habit" or "assumed nature of the task before us" may not be different at all; it, too, may be a case where people differ because they have different repertoires of assimilation paradigms. Neuman may be alluding to this when she writes:

> How can children's focus of attention be on "*abstraction*" and on "*the encapsulated object*", before they have experienced abstraction or encapsulation?

A computer created to deal with numbers can come to include also stored programs, which may be dealt with in many ways as if they themselves were numbers. In a similar fashion, strategies or task interpretations may be related to knowledge items that may, or may not, be present in a student's personal repertoire.

I am sure that it is clear to all of us that there is great complexity in the relations among whatever mental entities we choose to consider. I hope that our discussions will help us to acquire a deeper understanding of what lies beneath these important phenomena, and how we can best attempt to describe them.

## References

Alston, A., Davis, R. B., Maher, C. A., & Martino, A. (1994). Children's use of alternative structures. Lisbon: Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, vol. 2, 248-255.

Cochran, B., Barson, A., and Davis, R. B. (1970). Child-Created Mathematics. *Arithmetic Teacher*, 17(3), March, pp. 211-215.

Davis, R. B. (1984). Learning Mathematics: The Cognitive Science Approach to Mathematics Education. London: Routledge.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. Journal of Mathematical Behavior, 5(3), 281-304.

Dubinsky, E. (1991). Reflective Abstraction. In: D. O. Tall, (Ed.), Advanced Mathematical Thinking. Dordrecht: Kluwer Academic Publishers.

Fuson, K. (1982). An analysis of the counting-on solution procedure in addition. In: T. P. Carpenter, M. M. Moser, & T. A. Romberg (Eds.), Addition and Subtraction: A Cognitive Perspective. Hillsdale, NJ: Lawrence Erlbaum Associates.

Groen, G. J., & Resnick, L. B. (1977). Can preschool children invent addition algorithms? Journal of Educational Psychology, 69, 645-652.

Hatano, G. (1982). Learning to add and subtract: A Japanese perspective. In: T. P. Carpenter, M. M. Moser, & T. A. Romberg (Eds.), Addition and Subtraction: A Cognitive Perspective. Hillsdale, NJ: Lawrence Erlbaum Associates.

Miller, G. (1956). The magic number 7 ± 2. Psychological Review, 63, 81-97.

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA: author.

Neuman, D. (1987). The origin of arithmetic skills. A phenomenographic approach. Acta Universitatis Gothoburgensis. Göteborg.

Neuman, D. (1992). The influence of numerical factors in solving simple subtraction problems. Durham, New Hampshire: Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22, 1-36.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458-477.

# SUCCESS OR FAILURE IN ELEMENTARY NUMBER DEVELOPMENT

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*"Why do some children never develop arithmetic thinking?"* This is the important question Gray, Pitta & Tall have worked hard to understand during 10 years of empirical research. Their detailed description of differences in arithmetic and algebraic thinking between "low-" and "high achievers"– and their creation of the viable concepts *procedural and proceptual thinking*, and the *proceptual divide* – constitute a significant basis for future research on arithmetic and algebraic thinking.

In the paper presented here the authors have built a theory on cognitive styles on their empirical evidence. When I first read the paper, these theories made the empirical work, that I had earlier easily taken to my heart, look new, and very different. As a reactor to the paper I will use my opportunity to discuss some parts of the presented research, partly as I can see it if interpreted from the theories on cognitive styles, and partly as I can see it if interpreted from alternative views. To interpret other researchers' interpretations, however, is a delicate undertaking, and I will already now apologise for any fault that may occur, looking forward to seeing them corrected in our discussion at the research forum.

I have chosen three topics to discuss:

1) Relationships between cognitive styles and mathematical thinking
2) Questions concerning the origin of the proceptual divide
3) Pedagogical implications related to different views on the origin of numerical thinking

## Relationships between Cognitive Styles and Mathematical Thinking

In my attempts to understand the theory I first oscillated between two hypotheses on the nature of the relationship between cognitive styles and mathematical thinking:

1) Cognitive styles are what determine the kind of mathematical thinking that is developed.
2) Cognitive styles are developed in mutual interplay with mathematical thinking.

The second hypothesis was in line with my own view of knowledge as relations between an individual and the social-physical world, created in interactive activities, and constituting the referential base we use when we interpret and act on new phenomena (Marton & Neuman, 1996). According to this view content related knowledge and ways of experiencing how this knowledge can be learned is formed in mutual interplay. The character of the activities we carry out together is seen as greatly influencing our beliefs on the nature of the subject we learn about, as well as our beliefs on the course of action we may take to learn about this subject.

After several re-readings of the authors paper I arrived at the conclusion that this was *not* a view representative of the theories on cognitive styles. It was *the first hypothesis*, not the second one, that these theories were anchored in. Here are four examples from the paper of Gray et al. that finally lead me to that conviction:

> Our fundamental thesis was that different qualities of mathematical abstraction *were influenced by* the child's cognitive style.[1]

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[1] All italics in these four quotations from the paper of Gray et al are mine.

> The evidence seems to suggest that different cognitive styles *may lead to* the bifurcation in thinking that is evident in the proceptual divide
>
> ... we may provide the necessary support both to those who develop flexibly and also *to those who, at the very start of their mathematical development appear to be travelling along the wrong road*.
>
> Further, we conjecture that positive efforts to make the relationships implicit in proceptual thinking explicit to *those that do not have the associated flexibility* run the danger of being seen by some as a new set of procedural rules.

The quotations do not only underline the determining character of the cognitive styles. The researchers also express their conjecture that educational efforts to help "procedural" pupils think in a proceptual way are doomed to failure.

## Questions concerning the Origin of the Proceptual Divide

A prominent concept in the theory is "the proceptual divide", the bifurcation in the developmental line, where children's subsequent development of numerical thinking is said to go in different directions: towards success or failure. When children manipulate objects in early counting activities, the cognitive styles are thought to guide their perception in a way that decides the direction their development will take:

> It is our contention that *different perceptions of these objects*, whether mental or physical, *are at the heart of different cognitive styles that lead to success and failure in elementary arithmetic*.
>
> We conjecture that one cause of the proceptual divide is the qualitatively different focus of attention which, on the one hand places the emphasis upon concrete objects and actions upon these objects, and on the other on abstraction and the flexibility intrinsic within the encapsulated object.

How can children's focus of attention be on *"abstraction"* and on *"the encapsulated object"*, before they have experienced abstraction or encapsulation? I will return to this conjecture in the second of two subordinate questions on the origin of the divide:

1) When do the different kinds of numerical thinking first appear?
2) What is the goal that children intend to reach in different ways in their early counting?
    - What is the problem "low achievers" experience in their strive to reach this goal?
    - How is the goal reached by "high achievers"?

### *When do the two different kinds of numerical thinking first appear?*

To get a basis for the discussion – are the cognitive styles early developed characteristic features of the pupils, or are they developed in mutual interplay with a more or less fertilising pedagogy – it is urgent to know about when they first appear. The youngest children interviewed by Gray et al are 7 years old, and have already had instruction in arithmetic since they were 5. There are other researchers, however, who have studied children earlier in their school life.

Carpenter and Moser (1982), for instance, interviewed children three times during their first grade. They found that count back – a behaviour that is procedural as soon as it is related to double counting – was hardly ever observed if the children were allowed to use concrete aid or fingers. They conjecture "that some children never use a Counting-Down-From strategy prior to learning basic subtraction facts" (p 20).

An interesting longitudinal study of count back is also done by Svensson & Sjöberg (1981-82). It concerns the form of extremely laborious double counting backwards, where fingers are used for keeping track. Gray (1991) writes about this strategy:

> ... a substantial number of children use count-back to solve subtraction problems. This can be an horrendous process, particularly for the below average child. Indeed, watching such a child attempting to cope with 12 – 8 or 15 – 9 using such an approach, is a savage indictment of teaching methods that condone it (p 572).

Svensson et al interviewed 12 Swedish school children during their first three school years, giving them 66 subtraction problems within the number range 0–13 to solve at five different occasions: the second term in the first grade, and after that once every term until the second term in the third grade. In Sweden, children do not start school before the year they are 7, and no instruction in addition or subtraction is given at pre-school.[2] This study illustrated that "double counting backwards", with the fingers used for keeping track, did only appear in 8% of the answers in the investigation made in the second term in grade 1. In the first term in grade 2 it was used for 19% of the answers, and the second term in grade 2 the proportion was as high as 27%. By the first term in grade 3, however, many students seemed to have abandoned it. Only 16% of the problems were then solved by this method. The expectation would now be that it had been wholly abandoned in the last investigation performed in grade 3. However this was not the case. The proportion of problems solved by this method then still was as high as 14%. The conjecture put forward by Carpenter & Moser – that count back might not be used before children start school – is supported by this study. But it also supports the statement made by Gray et al: observations within any classroom show that all children will *not* encapsulate arithmetic processes.

A third example comes from a study where I interviewed 105 Swedish school beginners during their first weeks at school (Neuman, 1987, 1992). To solve 3 missing addend subtractions and one "take away" subtraction within the range 1–10, presented as word problems, none of these children used "double counting backwards", and "double counting forwards" appeared only in 10 answers. Also these observations support the hypotheses that counting back carried out as double counting is not an informally learned method. However, I also observed many examples on proceptual thinking in this study. As many as 154 answers given by these 105 7-year-old school beginners to these four subtraction tasks were of a kind that by Gray & Tall (1994) is categorised as expressions of a proceptual behaviour.

To conclude, these studies indicate that

1. No children seem to be "travelling along the wrong road" before school start.
2. Many children illustrate that they develop proceptual thinking before they begin school.

An alternative theory, than the one on cognitive styles as determining success or failure, stands out against these observations. All children might to begin with follow a developmental track, natural for children who take part in activities representative of our culture. Some children might get lots of experiences of mathematics during their pre-school years, and begin during these years informally to develop proceptual ways of thinking. Others might not even have understood the cardinality principle in concrete counting activities at this point. While the former group of children may not care a bit about methods introduced at school, these methods might *form* the way children in the latter group think about what it means to add and subtract. Instead of referring to a bifurcation, and a divide in thinking, we may, according to this hypothesis, refer to a *turning from the main road*, ending up in a cul de sac. Children who do not develop ideas of addition and subtraction in informal activities are forced into this turning by the pedagogy they meet at school.

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[2] This has changed a bit during the last two years, where some pre-school teachers might start with this kind of instruction in the "6-year-old groups" that often are formed now-a-days.

However, a course of action is implied to reach a goal. The goal that implies respectively procedural and proceptual acts, and the problem arising to "procedural children" trying to reach this goal, is what I now will turn to in my discussion.

*What is the goal that children intend to reach in different ways in early counting?*

What children experience as the goal for early addition is to become aware of the numerosity of the whole, that is constituted by two added sets. In subtraction it is to become aware of the numerosity of one of the sets that constitutes the whole. To reach this goal, however, they must encapsulate several procedures.

Fig 4 in the paper of Gray et al illustrates that the researchers see the proceptual divide as appearing when children have begun to count on, something that does not fit well with the conjecture that inclination to encapsulate procedures should be a characteristic of the proceptual style only. When children count on, they have encapsulated the procedure related to the first of two added sets. The difference between the two "styles" rather seems to be related to children's success or failure in their attempts to encapsulate the procedure related to the second set. Some children never encapsulate this procedure, not even for all combinations within the range 1–10 (Gray & Tall, 1994, fig 4 and 5, Neuman, 1987). What makes encapsulation of the procedure related to this set more difficult than the one related to the first set? The answer is that encapsulation is not of the same character in these two contexts. For children who experience a symbolism of the kind 2 + 9 as an inclination to carry out a procedure, the encapsulation of the second set becomes extremely problematic.

*What is the problem "low achievers" experience in their strive to reach the goal?*

I will illustrate their problem, using a symbolism embedded in a word problem as an example: "You have 4 glasses but there are 10 children who want to drink; how many more glasses do you need?" Young children experience subtraction problems of this kind as "missing addends" (Carpenter & Moser, 1982), as 4 + _ = 10 in the example. The children who have begun to count on understand that the last word in the numeration that is possible to carry out for the first set denotes the numerosity of this set. The last word in the numeration done for the second set, however – "ten" in the example – does not denote the numerosity of this second set. If the number of words listed in the count on procedure is as large as it is in this problem, it is not either possible to subitise it.[1] There is no intuitive and instant way at all that makes it possible to experience the numerosity of the second set, if it is outside the subitising range. Similar difficulties are related to symbolisms of the kind 4 + 6 and 10 – 6, but there the problem is to know when to stop the count on/back procedure.

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[1] I use the word "subitise" here, in the meaning "experience without counting".

To reach the goal – to find the symbol that denotes the numerosity of the second set – children must invent an idea or a method. While the "high ability children" invent *a lot of different ideas* (Gray & Tall, 1994, Neuman, 1992) the low ability children invent *one method*: they double count. Double counting is not a procedure possible to encapsulate, and the success of "high achievers" does not depend on some mysterious gift to encapsulate this kind of procedures. It depends on that they begin to create more viable methods before they meet the instruction at school. Children who have learned a procedural behaviour through this instruction, however, can never learn the facts they are supposed to know through encapsulating their double counting.

In their rejection of the denotation "slow learners" for "low achievers", Gray & Tall (1994) underline that "low achievers" do not learn the same techniques as their class mates at a slower pace. They learn *different* techniques, techniques forcing them into a cul de sac. This is an assertion that exactly pictures the hopeless situation of the "low achievers", one I fully agree on. Yet, our opinions on a theory explaining the origin of this hopeless situation are very apart from one another.

*How do "high achievers" succeed in reaching the goal?*

I have questioned the assumption that success or failure should depend on cognitive styles inclining children to put the emphasis on respectively actions or abstractions in their early manipulation of objects. Yet, I will stick to the idea that the manipulated objects are of importance. The character of these objects – not cognitive styles – might be what helps children to, or prevents them from, encapsulating procedures.

Several different kinds of objects are used in this context, as counters, unifix cubes, or sticks. But the objects that are used can also have a structure. Treffers (1991) recommended the use of an "arithmetic rack" with a combined 2- and 5-structure – two rods with 10 marbles, five red and five blue. Japanese children use another kind of structured objects, TILES, rods with a semi-decimal structure: 5-rods and single TILES (Hatano, 1982). And most children use their own fingers – either their teacher encourages or forbids that – also these with a semi-decimal structure.

Japanese teachers found, Hatano (1982) underlines, that five as "an intermediate higher unit" was

> necessary when they were trying to teach numbers to young children and retarded children. The use of five as an intermediate unit, adopting unsegmented TILEs of five instead of connected or isolated singe TILEs, makes it possible ... to grasp them (numbers within the range 1–10) almost intuitively ... (p 216, my bracketing).[4]

Referring to Samejima & Hatano and to Ito (in Hatano, p 214), Hatano adds:

> Only after numbers were associated with the corresponding TILEs or their mental representation could most young children use cardinal numbers (p 214).

According to the observations made by the Japanese teachers and researchers, encapsulation of procedures does not occur if unstructured material is used, while it occurs almost intuitively if the material has a structure with an "undivided five".

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[4] The children do themselves cut these TILES and they choose themselves when they want to use the unsegmented five.

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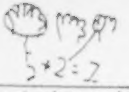

Fig A

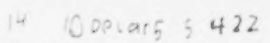

Fig B *The child writes: 14 10 share 5 5 4 2 2*

Fig A above is drawn by a girl in grade 2, asked to write something, or to make some kind of drawing, that illustrates how she solved the problem: "Mia has 14 mushrooms in her basket. Måns has half as many. How many does Måns have?"". The girl has drawn her image of 14 as 2 hands and 2 + 2 fingers. Her answer – "Seven" – is easily reached without counting, since one of her images of seven – 1 hand and 2 fingers – is included in her known pattern for 14. The same image is explained in a bit more mathematised way in fig B.

In the pictures illustrating the images of 6 + 3 reported by Gray et al (fig 5) the rectangular 9-dots pattern is of the same character as the picture of the hands and fingers in fig A: the two parts – 6 and 3 – are known patterns within the directly recognised pattern "3 threes makes 9".

All patterns, however, are not structured in a way that is generally useful. The 9-dots pattern, for instance, that was of great support when the problem 6 + 3 (fig 5) was solved, does not help the child who solves the problem 9 – 5 (fig 6) to avoid double counting backwards, since it does not include the dice pattern that is 5 for this child.

If the girl who drew the picture in fig A would solve the problem 9 – 5, on the contrary, she would immediately know the answer four, since one of the patterns she knows for 5, the hand, and one she knows for four, the hand minus one finger, is included in one of the patterns she knows for 9, the hands minus one finger. Yet, this girl would as easily solve the problem 6 + 3, since one of the patterns she knows for 6, the hand plus one finger, is also included in the pattern for 9 constituted by her hands. With the help of the fingers – used concretely, as images or as abstractions of the kind mirrored in fig B – children can almost intuitively experience all those number combinations within the range 1–10, where one of the sets is 5 or larger. This is exactly as when TILES are used. When children use their fingers, however, they can experience also those combinations where both sets are smaller than 5, moving one thumb from one hand to the other, for instance, to transform 5|2|7 to 4|3|7. With the help of the fingers used in this way, children can directly – without any need of counting – experience all numbers with their constituent parts within the range 1–10 (except for 3|3|6). This was the way we once could think of number combinations when we used the ancient Roman numerals, that pictured our hands (fig D). When fingers are used in this way – and not for keeping track in double counting – the relation between addition and subtraction also becomes self evident (fig E). Encapsulated knowledge of subtraction within the range 1–10 as a network of relations is a necessary presumption for addition and subtraction within higher

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The drawings are made by children in a pilot testing to the Swedish diagnoses in mathematics for grade 2.

A presumption for that is that the first hand is kept undivided. Several Swedish school starters avoided intuitively to divide up this hand when both sets were 5 or larger. Several of them had also spontaneously found the idea to move one thumb between their hands when both sets were smaller than 5. These children used their fingers as"finger numbers" (Neuman, 1992) ,concretely or as images, in the way Brissiaud (1992) refers to as "finger symbol sets", where 5 plays a major role, and where the fingers from five and on end at the thumb, the second forefinger, long finger, and so on.

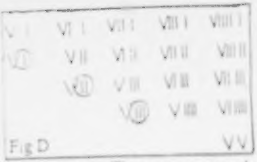

Fig D

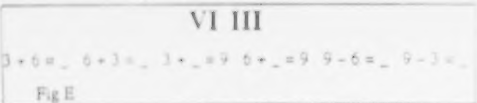

Fig E

number ranges. To carry out the addition 27 + 8, for instance you have to know that the missing addend in 7 + _ = 10 is 3 and that 8 – 3 = 5. Then you can easily see the problem as 30 + 5 = 35, and also estimate the sum of 27378 + 8269. In the same way you can see 32 – 8 as 30 – 6 = 24, as far as you directly know that 8 – 2 = 6 and that 10 – 6 = 4, and you can also make a rough estimate of what 32475 – 8223 makes.

## Educational implications

I have used the example of fingers to illustrate ways that can be used informally to create proceptual thinking. "Informally" does not mean without support from the surrounding world. Children have learned to count from this world, to understand that the early subitised 2- and 3-groups are related to the words "two" and "three", to illustrate their age with their fingers and to relate number words to dice and domino patterns, already before they have counted the finger symbols and the dice patterns. Bit by bit they then relate those experiences to each other. They may, for instance, in some situation have counted the 2- and 3-groups. Then, as Wynn (1992) says, they may make the analogy between their own "magnitudinal relationships" and the "positional relationships of the number words" (p 250). In similar ways they might have felt the need to count what Brissiaud (1992) calls the early known "finger symbols sets" (p 47), thus relating an "analog system" and a "logico-mathematical system"(p 65) to each other. In such situations, they understand the cardinality of the last number word in their ordinal counting procedure since it is the one they earlier have used to denote their not counted cardinal finger symbol sets or 2 – 3 groups.

That those ideas are formed outside school does not mean that it is impossible to organise formal teaching in a way, where they can be invented. There are many examples of mathematics classrooms, where children structure objects in contexts of meaningful play and games, and where they in interactive problem solving explain and argue for what they do, getting the opportunity together to explore the structures they have formed (e. g. Yackel & Cobb, 1996, Neuman, 1987, 1993).

The educational implications proposed by Gray et al are that "we may provide the necessary support both to those who develop flexibly, and also to those who at the very start of their mathematical development appear to be travelling along the wrong road". Yet, the researchers are hesitant, to efforts aimed at making "the relationships implicit in proceptual thinking explicit to those that do not have the associated flexibility". Gray (1991), for example, expresses "a degree of scepticism in claiming that benefits arise from the protracted use of structural apparatus" (p 572), and he is critical to children's use of "various finger configurations to represent different numbers", since that might lead to "short set success with small numbers but considerable difficulty, and even failure, with larger numbers." (p 572). What he recommends is that below average children should be

> given an opportunity to expand their experience of seeing problem and product without the clutter of a procedure. Graphic calculators, which enable a simultaneous display of a problem and its product, may provide such an opportunity (p 572).

Gray et al have with great power illustrated how hordes of children are tormented by the demand to do hundreds of sums without ever understanding why they do them and with a gradually fainting and never realised hope that they finally will learn the facts they are supposed to learn through this training.

Their research could be a most powerful instrument for a reform movement concerning introduction of mathematics at the primary level. However, the interpretations they make of their research rather lead to the thought that the support we can give to those who fail should be to teach them how to live with their handicap. I do not believe that this is the message the researchers have in mind. But it is hard to find an alternative in their writings.

I have here tried to outline some more positive alternatives to the interpretations possible to make of the research Gray et al present. There are probably many more alternatives that will be proposed at the research forum. To help all pupils find the joy inherent in a proceptual thinking during their first years at school is an urgent challenge to all researchers interested in mathematics education at primary level. It is also a question that concerns equal right for all children to realise the seed of numerical ability sown in all of them long before school start.

**References:**

Brissaud, R. (1992) A toll for number construction: Finger symbol sets. In J. Bideaud, C. Meljac and J.-P. Fischer (Eds) *Pathways to number. Children's developing numerical abilities*, pp 41–66. Hillsdale, N J: Lawrence Erlbaum.

Carpenter, T. P. & Moser, J. M. (1982). The development of addition and subtraction problem-solving skills. In Carpenter, T.P., Moser, J.M. & Romberg T.A. (Eds.), *Addition and subtraction, a cognitive perspective*. Hillsdale, N. J. : Lawrence Erlbaum Associates.

Gray , E. M. (1991) An Analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics*, 22, 6, 551 – 574

Gray, E. M. and Tall, D. O (1994) Duality, Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for research in mathematics education*, 25, 2, 115-141.

Hatano, G. (1982) Learning to add and subtract: a Japanese perspective. In T P Carpenter, M M Moser and T A Romberg (Eds) *Addition and subtraction. A Cognitive Perspective*. Hillsdale N J: Lawrence Erlbaum Associates.

Marton, F. and Neuman, D. (1996) Phenomenography and children's experience of division. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin and B. Greer (Eds) *Theories of mathematical learning*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

Neuman, D. (1987) *The origin of arithmetic skills. A phenomenographic approach*. Acta Universitatis Gothoburgensis. Göteborg.

Neuman, D. (1992) The influence of numerical factors in solving simple subtraction problems. In *Proceedings of PME 16. Psychology of Mathematics Education*, Durham, New Hampshire.

Neuman, D. (1993) Exploring numbers through subtraction in primary mathematics. In T Breitig, I Huntley & G Kaiser-Messmer (Eds) *Teaching and learning mathematics in context*. New York: Ellis Horwood.

Svensson, O. & Sjöberg, K. (981/82) Solving simple subtractions during the first three school years. *Journal of experimental education*, 2, 91 – 100

Treffers, A. (1991) Didactical background of a mathematics program for primary education. In L. Streefland (Ed) *Realistic mathematics education in primary school. On occasion of the opening of the Freudenthal Institute*. Utrecht: Freudenthal Institute.

Wynn, K. (1992) Children's acquisition of the number words and the counting system. *Cognitive Psychology* 24, 220-251

Yackel, E. & Cobb P. (1996) Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27, 4, 458–477.

**Research Forum**

## RESEARCH ON THE FUNCTION CONCEPT

| | |
|---|---|
| *Coordinator* | Gard Brekke |
| *Presentation* | Unifying cognitive and sociocultural aspects in research on learning the function concept<br>by Rina Hershkowitz and Baruch B. Scharz |
| *Reactor* | Joel Hillel |
| *Presentation* | Emergence of new schemes for solving algebra word problems<br>by Michal Yerushalmy |
| *Reactor* | Joao Filipe Matos |

# Unifying cognitive and sociocultural aspects in research on learning the function concept

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Baruch B. Schwarz — The Hebrew University

*This paper presents two studies on learning the function concept which are part from a research and development project. In the first study which is based on a questionnaire, we characterized the concept image of Grade 9 students learning in an interactive environment based on multi-representational tools and open-ended activities. We found that the students (a) used many examples (b) provide rich justifications to their answers (c) showed flexibility within and among representations (d) considered the acceptability of answers in the light of the context (e) integrated prototypic examples with other examples. The second study was a classroom research with the same subjects. This study took into consideration interactions between individuals, the problem situation, tools and community. We observed actions, their objects and the artifacts that mediated their construction and their modification during the phases of an activity. Tracing the modification of actions led to a study of concept learning processes within an interactive environment. At the end of the paper, an attempt is made to unify the findings of the two studies.*

## Introduction: The research and development process

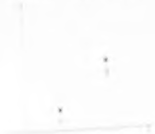
This paper focuses on a series of studies on learning the function concept. They are part of a research and development program initiated twenty years ago, that took place in a number of cycles. Each cycle had its own novelty, but was developed on the cumulative experience gained from the previous ones, on the development of theoretical frames of the team, on the availability of new tools (including computerized tools), and on the research undertaken in the previous cycles.

The development component in each cycle consisted of: (i) textbooks for students and teacher guides; (ii) in-service teacher education; and (iii) the implementation of a program on a large scale (Dreyfus, Hershkowitz & Bruckheimer, 1987). The first cycle (which finished about seven years ago) was quite traditional: sub-concepts, representations, and procedures were presented and exercised linearly, the formal definition of function and its sub-concepts being introduced quite early. The teacher generally led students in a prescriptive way in a classroom forum.

The second cycle took into consideration research findings: Vinner & Dreyfus, (1989) showed that university students do not refer to abstract definitions to decide whether a graph represents a function or not, although they are taught functions through abstract definitions. Karplus (1979) used questions such as the "bacteria puzzle" (labeled I3 in Fig. 1), to show that more than 40% of high school students used linear inter/extrapolation to find bacteria population at different temperatures, even when the results made little sense (i.e. a negative number of bacteria).

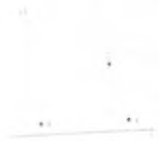
Markovits (1982) who studied first cycle students, confirmed Karplus' findings. She also added purely formal mathematical items (see items I1 and I2 in Fig. 1) to find that all Grade 9 students chose a linear function to exemplify a function passing through two given points (I1a), and about half of them declared that only this function fits. Only a third declared that an infinity of functions fits. More than half of the students thought that no function passes through three non-collinear points (see I2a in Fig. 1). The others generally chose a quadratic function.

I1a. Draw the graph of a function such that the coordinates of points A and B represent the preimage and the corresponding image of the function.



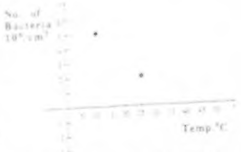
I1b. The number of different such functions that can be drawn is:
(a) 0 (b) 1 (c) 2 (d) more than 2 but less than 10
(e) more than 10 but not infinite (f) infinite
Explain your answer.

I2a. Draw the graph of a function such that the coordinates of points A, B and C represent the preimage and the corresponding image of the function.
I2b. The same as I1b.



I3. A scientist undertakes a study on bacteria cultures. Bacteria species are known to live at different ranges of temperatures; therefore the number of bacteria in such a culture depends on temperature.

The number of bacteria at the temperatures of 10°C and 25°C is marked in the following coordinate system.



a) At the next step of the study, the scientist needs to know the number of bacteria at 20°C. What can you say about the number of bacteria at that temperature? Explain your answer.

b) The scientist wants to predict the number of bacteria at 30°C. What can you say about the number of bacteria at that temperature? Explain your answer.

c) The scientist wants also to predict the number of bacteria at a temperature of 45°C. What can you say about the number of bacteria at that temperature? Explain your answer.

Figure 1

The curriculum in the second cycle took these findings into consideration and presented more examples of functions (to avoid a "prototypic concept image"), students were encouraged to work at times in an explorative mode and most of the activities were followed by sessions of consolidation.

The development in the third and latest cycle, "the Function Project", was based on research done on the role of multirepresentational software in learning the

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function concept (e.g., Kaput, 1992). In a study undertaken towards the beginning of the third cycle, Schwarz and Dreyfus (1995) showed how the manipulation of computer partial displays of graphs and tables (called representatives) helps inferring properties of functions. The Function Project proposes an introductory course for Grade 9 that conforms to the official syllabus in Israel. It consists of a chain of tasks around problem-situations whose contextual frame is thought to facilitate the growth of the function concept. Students in the experimental classes have tools at their disposal such as graphic calculators or multirepresentational software. The main characteristics of the project are:

- Students are encouraged to take their own decisions about which representation/s to use, when and how to link them, and in which medium to work, in order to work on the problem situation.
- Most of the inquiry is done in small groups in a collaborative mode.
- Students are asked to write group or individual reports, in which they are encouraged to compare, and critique the steps they and others went through. They are specifically asked to report on their solution processes, for example on how and why an hypothesis was raised or discarded and why another one was preferred.
- The teacher plays several roles, being alternatively a facilitator, a modeller and a coordinator of debates around verbal reports and critiques in the synthesis session with the whole class. Such syntheses provide opportunities for students to report on, evaluate and criticize their collaborative or individual work in a public forum.
- A crucial part of the didactic engineering within this environment is expressed in the task structure. The tasks around problem-situations consist of up to five phases.

Our purpose in this paper is to present the research in the third cycle according to two different approaches: (a) "Classical" cognitive research aimed to characterize students' function concept image at the end of the year long course. The curricula of both the second and third cycles are still in use in Israel, so we could contrast the nature of the concept image in the third cycle with the previous cycle. (b) A classroom research aimed at tracing changes in practices and actions during the phases of an activity. During these phases, students solved a problem, reported on it, criticized, and reflected on their or others' actions. In fact, changes evidence concept learning processes as they appeared in the interactive environment.

## The first study: the nature of the function concept image

Researchers in mathematics education have differentiated the concept as it follows from its mathematical definition from the concept image as it is reflected in the individual mind (Vinner, 1983). Vinner and Hershkowitz (Vinner & Hershkowitz, 1983; Hershkowitz, 1989) studied concept learning in geometry and found that prototypical examples play a crucial role in learning geometrical concepts. Similarly

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to Rosch and Mervis (1975) research on natural categories, they found that prototypical examples are usually the subset of examples that have the "longest" list of attributes -- all the critical attributes of the concept (those attributes that an instance must have in order to be a concept example), and self-attributes (those attributes that only the prototypical example has). As the prototypes are the examples of the concept that are perceived first, it was found that often children's concept image includes mainly the prototypes. This explains why students tend to identify a concept with one or few prototypical examples, and often, the concept image develops from a unique prototype to include more examples whose distance from the prototype increases progressively, rather than from its formal definition. It was shown that concept learning in geometry is governed by prototypes (Hershkowitz, 1989), their role being beneficial or detrimental, depending on their (im)proper use as frames of reference in the judgment of other examples.

It seems that the above interpretations apply also to function concept learning. Or at least, Karplus, and Markovits findings raise the hypothesis that students' concept image is prototypical, where linear and quadratic functions are prototypes. And indeed linear and quadratic functions have a long list of self-attributes in addition to their critical attributes as examples of functions. The studies by Karplus and Markovits seem to show the prevalence of the linear function as an almost exclusive prototypical example. It is only when a linear function does not satisfy the given conditions, as in the case of the three points (I2a in Fig. 1), that students move to a quadratic function.

In the present study we check the above hypothesis. We also have investigated concept image in the new interactive environment (the third cycle). Two groups of Grade 9 students, $G_2$ and $G_3$, belonging to the second and third cycles respectively, underwent a one year long course. A questionnaire was administered io both groups at the beginning of Grade 10. Few items from the questionnaire are analyzed here quantitatively as well as qualitatively. Conclusions on the concept image in the two groups are then discussed.

### *The function concept image: Richness, linearity and prototypicality*

We borrowed the first items (I1, I2 and I3) from Karplus and Markovits studies (see Figure 1). In Item I1a, more than half of $G_2$ and $G_3$ students choose to exemplify a function passing through the two given points by the linear function only. For 17% among $G_2$-students and for none among $G_3$-students, the linear function was an exclusive example. We can conclude that for many students in both groups (more in $G_2$ than in $G_3$) the linear function was "popular", but not meaning that it was considered as exclusive. For all $G_3$-students and for most of $G_2$-students, passing through two points is not limited to linear functions. Thus, students in both groups are far less "linear" than in the Markovits' study, in which more than half of the students thought that the linear function was the only one to pass through the two given points.

In the case of the three given points (I2), both groups overcame "Markovits students". The difference between $G_2$ and $G_3$ is expressed mainly by the number of idea units in justifications, the number of examples, and their nature: $G_2$-students used 1.17 examples on I1a and 1.97 idea units to justify them (in I2b), whereas $G_3$ -students used 1.56 examples and 2.56 idea units. The following examples demonstrate the nature of justifications and examples that were given:

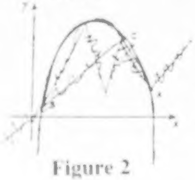

Figure 2

Example 1: A $G_2$-student failed in his attempt to draw linear functions through the three given points (I2a), so he "erased" his linear attempt and drew a quadratic function (Fig. 2). He stated that there is only one function in I2b and explained: *because one cannot draw a linear function here.*

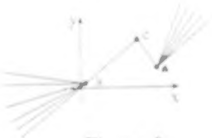

Figure 3

Example 2: A $G_3$-student also exhibited a "linear concept image" but in a quite different way. She drew a family of piece-wise linear functions (Fig. 3), declared that there is an infinity of functions through the three given points (I2b) and explained: *there is an infinity of possibilities to continue from points A and B as I drew in the graph.*

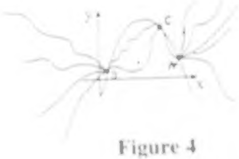

Figure 4

Example 3: Another $G_3$-student drew many examples (I2a), without any linear elements (Fig. 4), said that there is an infinity of functions (I2b), to add: *It can be one graph in the domain, and it varies in infinite ways outside. Also, it can change in the domain if I wish it (as long as there are not two images for one pre-image). See my drawing!*

The justification in example 3 exemplifies a judgment based on the definition of the function concept, in contrast with the prototypical judgment of the $G_2$-student in Example 1, who chose a quadratic function just because he could not find a linear one.

These examples clarify our methodology of counting idea units in justifications. The justification in Example 3 contains three idea units: (1) it varies in infinite ways outside; (2) it can also change within the domain; (3) there are not two images for one pre-image. The first example contains one idea unit only.

The results in I3 express clearly the impact of the context: About half of $G_2$-students gave a justification based on linear interpolation or extrapolation for I3a, I3b and I3c and only about 18% gave a justification based on the context. About 40% of $G_3$-students used linear justifications in I3a and I3b and this proportion dropped to 34% in I3c (all students who used linear extrapolation for I3c, reached a negative number of bacteria), but more than half gave contextual justifications for the three items. Thus, far more $G_3$-students took the context into consideration, and even when they used linear attributes or strategies, they considered whether they match the context, as shown in the following example.

Example 4. A typical "linear" $G_3$-student answer to I3a is:

*You can say that the number of bacteria is close to 3, because at* $10^{o}$*C they were 5 and at* $25^{o}$*C they are 2, so when the temperature rises, their number decreases, so it will be between 5 and 2. At* $20^{o}$*C, they are close to 4 (between 5 and 2).*

But then, this student drew an atypical curve (Fig. 5) and added:

*All answers are based on the assumption that when the temperature rises, the number of bacteria decreases. But maybe it's a special kind of bacteria where the function is the following* (Fig. 5) *without any proportional relation between temperature and number of bacteria. (For example at* $10^{o}$*C there are 5, at* $15^{o}$*C their number rises to 10, and at* $25^{o}$*C there are only two left).*

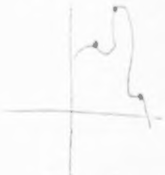

**Figure 5**

In this example, the student clearly states that she chooses a linear interpolation when it is based on an assumption she considers as quite reasonable, although she is aware that there are other possibilities. And indeed, this student, like most $G_3$ students does not extrapolate in a linear way the number of bacteria in I3c.

In summary, similarly to the students in Karplus' and Markovits' studies, $G_2$-students often based their justifications on the self-attributes of the linear function, even when this led them to aberrations. Such an extrapolation is an example of the imposition of a self-attribute of linear functions on situations where it is not appropriate. In contrast, when using linear strategies or attributes, $G_3$-students took the context into consideration. In I1a, there is no reason not to use a linear function to exemplify the function attribute. In I3c where a linear function is inadequate, the student searches for other kinds of functions matching the constraints of the problem. Thus, for $G_3$-students, linear inter/extrapolation is a legitimate strategy applicable to many situations (not necessarily linear), but these strategies are judged against the reasonableness of the results.

### *Understanding attributes*

We consider the matching between representatives from different representations as a "measure" of understanding. Items I6 and I8 are such tasks. Their description shows that they demand the understanding of attributes of functions. I6 (Fig. 6) checks whether students are able to recognize that graphs 1 and 4 are representatives of the function $f(x) = -2x^3$, and that graphs 2 and 3 are not.

I6. Given the function $f(x) = -2x^3$. Which graph cannot be a part of the graph of f? Justify your answer.

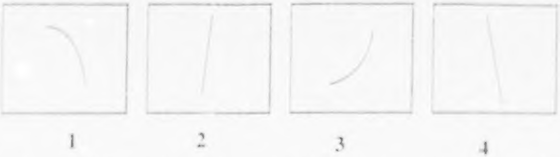

**Figure 6**

Students who have difficulties in recognizing representatives as being malleable objects representing the same entity, will infer that graphs 2 and 4 cannot be a part of the function because they "are" linear and the function f(x) is not. Such students extract a wrong attribute (linearity) and fail to match the right representatives with the "formula". I6 is easily solved by noticing that the function f(x) decreases.

Example 5 shows a typical justification in which a $G_2$-student fixes her judgment on the form of the graph and shows the prototypical status of the quadratic function:

*Graph 1 fits because the graph has a parabola with a maximum because it's $-2x^3$ (minus). So, it is going down from the extremum--maximum* .

In contrast $G_1$-students use more sophisticated examples in a more flexible way as shown in Example 6:

*Graph 1 can fit the function, because when you have a function $x^n$ with n odd, the function looks like this* (Fig. 7a) *and when you multiply it by a negative number the function will be in the other direction* (Fig. 7b). *So graph 1 can fit the part of the graph that I marked. Also, that kind of function always decreases, which can be seen only in graph 1.*

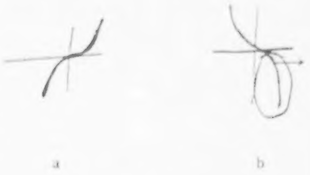

**Figure 7**

This example shows also another typical phenomenon. Justifications are compound (even when they are partially wrong). This justification considers transformations on the graph. Judgment is initiated by reference to the function $y=x^n$ with n odd. The student then reconstructs the graph of the function $y=-2x^3$ from this reference by linking the multiplication by a negative number with a transformation of the initial graph. In this sense, knowledge about odd functions is derived from a prototypical

typical, and indeed 53% of $G_3$ students responded correctly, very often by reconstructing the graph from the graph of the function $y=x^3$. In light of the results of I6, it appears again that $G_3$-students justify their answers more, and are able to analyze attributes of graphs without attaching them to possible prototypes. They use compound justifications in contrast with short justifications used by $G_2$-students to link to prototypes. It would seem, therefore, that $G_3$-students are more able to understand attributes through representatives.

The last item I8 shows this conclusion even better.

Item I8 requires the match of a graphical representative a line bisecting the angle between the axes (without any indication of units), with possible algebraic formulae (Fig. 8). The line can represent any function of the form $y=ax$. But students who declare that such a bisecting line represents the function $f(x)=x$ only, cannot extract the only relevant attribute of the graphical display, namely, it represents any (increasing) straight line passing through the origin of the axes.

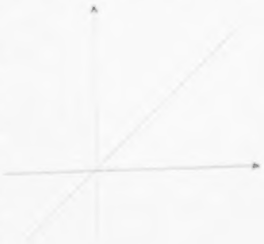

**Figure 8**

To answer this question, one needs to recognize that a straight line passing through the origin belongs to the family of functions $f(x)=ax$. This attribute is invariant under the change of units on the axes, although the representative may represent different linear functions. The results on this item are telling: 37% of $G_2$-students as against 66% of $G_3$-students answered correctly that the graph can represent an infinity of functions. 61% of $G_2$-students as against 31% of $G_3$-students concluded that the function is $y=x$. Here also, $G_3$-students integrated more idea units, as shown in Example 7:

*The graph is straight and passes through the origin so it's "ax" but "a" could be any number and we don't have units on the axes, so it's impossible to know. You know only that "a" should be positive because the function increases.*

## *Concluding remarks.*

We found that $G_3$-students (a) use more examples (b) justify better (c) show more flexibility within and among representations (d) consider the acceptability of answers in light of the context (e) better integrate prototypes with other examples. $G_3$-students like $G_2$-students exhibited prototypical elements in their function concept image. But, for $G_2$-students, these prototypes often acted as "brakes" on concept learning. In contrast, $G_3$-students' use of prototypes was very often beneficial, either to exemplify an attribute because of the simplicity of the prototype or to construct another example because the prototype is a well-known reference to begin with (for

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example the use of the graph of the function $y=x^3$ to find the graph of $y=-2x^3$). For some $G_3$-students, prototypes acted as levers for concept learning.

The interpretation of the difference between concept images of students studying curricula in the second and the third cycles left many issues unresolved. For example: Why were justifications better articulated in the "third cycle"? How did conceptual change take place over the year? or What was the role of the enculturation of the class to reflect upon their own or others' actions in conceptual change? What was the role of computerized tools in this conceptual change? These questions were investigated in the classroom research study.

## The classroom research study

The general theoretical perspective we adopted integrates cognitive development with social interaction and practices. Other researchers have adopted this approach (e.g., Lampert, 1990; Yackel & Cobb, 1996). The central part played by tools in our classroom activities led us to adopt a perspective based on Vygostskian ideas, and particularly appropriate to rich environments, the Activity Theory (Leonte'ev, 1981). This theory is a framework for studying different forms of practices as development processes, with both individual and social levels interlinked at the same time. Activity Theory is a descriptive tool rather than a predictive theory (Leonte'ev, 1981). It claims that consciousness is located in everyday practice. The understanding of the individual, other people, and artifacts in everyday activity is the challenge Activity Theory has set for itself. One key idea of Activity Theory is that the unit of analysis is not human action (as in laboratory experiments) but the activity, that is a minimal meaningful context for individual actions. Activities have their own history, parts of older activities often being embedded in the current activity. They always contain various artifacts (tools, signs, forms of work organization, etc.) through which the elements of the activity are mediated by the subject (individual or community). They consist of chains of actions (cooperative or individual) related by the same object and motive, but these actions can be understood only within the activity in which they are embedded. Each action is accomplished by carrying out operations that are often not made explicit by participants and/or automated. The three levels of analysis (activity, actions and operations) were used here for studying concept learning.

The research was based on the observation of several activities organized around problem-situations during the year within the same classroom (with $G_3$ -students), on the analysis of students' group reports and on interviewing the teacher before and after each observed activity. The observations were documented either by observers taking notes, by video, or both. Conceptual change was checked in micro genetic (change within and between the phases of the same activity) and ontogenic (change between activities) analyses. Our perspective here is micro genetic: we focus on one activity named "Overseas" given toward the end of the year. It consisted of four phases. In Phase 0, students were given the following homework assignment:

*The freight company "Overseas Inc." uses containers to ship goods by sea from country to country. The containers are big boxes made of wood. Their base needs to be a square, and their volume, must be 2.25 m*$^3$ *(Fig. 9). The containers must also be open at their top. Can you find two or three examples of such containers?*

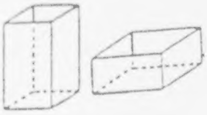

Figure 9

*Find the dimension of two containers and construct suitable paper models (1m of the container = 5 cm of the paper).*

*Calculate the wood area needed to construct each of the two containers.*

Students brought their models to class and were given a worksheet:

*As wood is expensive, the company is interested in designing ideal containers with as little wood as possible.*

*Try to make hypotheses on the dimensions of the ideal container; explain your steps.*

*Can you help the company find the dimensions of the "ideal container"?:*

In Phase 1, the students were invited to work in groups of four. Ten groups were formed, each of them being equipped with their models and two graphic calculators. Then, each group was asked to write a report on their ideas and steps (Phase 2). Finally, the teacher initiated a class discussion in order to report and criticize in a class forum. Phases 1-2-3 lasted about 90 minutes. The analysis of this activity is mostly based on one group of students. Typically, reaching the accurate solution of Overseas was not the main interest, and students were more involved in an activity, that is, in a setting in which they constructed meaning together. This paper focuses on Phase 1, in which we observed a four girls group, and on Phase 3.

### *Phase 1: Solving Overseas*

The actions and objects in Phase 1 served two goals. First we intended to locate and characterize the development of objects and actions formed during problem solving. Second, all problem-solving actions were anchors for ulterior phases (when reporting or criticizing).

The students brought to the class paper models of containers (1m=5cm) with a given volume (2.25 m$^3$). We focus here on a group of four girls. They tried first to push collaboratively the limits of what could be guessed or hypothesized, before using computerized tools. They pursued these actions for almost all the available time, leaving only a few minutes for successfully drawing the graph with the calculator and reading the minimum at the very end of their collaborative work. Such an allotment of time was legitimate and fitted the enculturation of this class to value processes and ways rather than results. Figure 10 is a schema of the dialectic process that the four students went through during Phase 1:

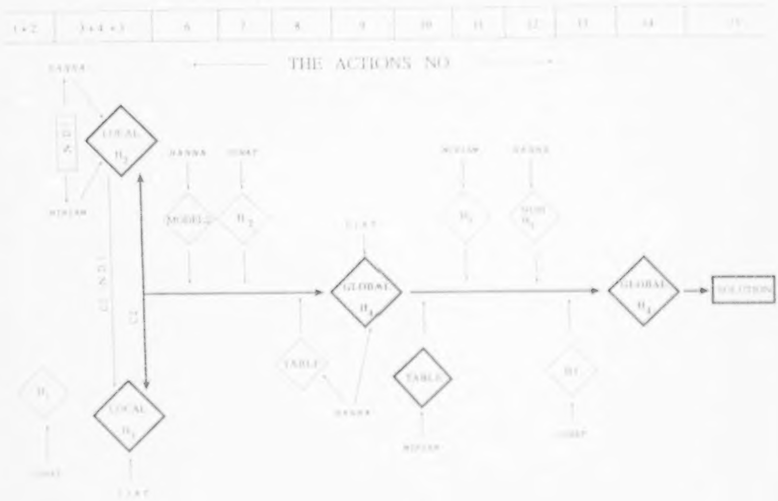
1+2
3+4+5
6
7
8
9
10
11
12
13
14
15
THE ACTIONS NO
HANNA
LOCAL
$H_2$
MIRIAM
HANNA
OSNAT
MODELS
$H_2$
C1 AND C2
C2
LIAT
GLOBAL
$H_4$
MIRIAM
HANNA
GLOBAL
$H_4$
SOLUTION
TABLE
HANNA
TABLE
MIRIAM
$H_1$
OSNAT
LOCAL
$H_3$
LIAT

Figure 10

The four girls began the session by computing the measures (side, height, and overall surface) of the models. Then Osnat raised the hypothesis that *"because the height of the container is multiplied by four, the minimum surface is attained when the height is the smallest possible"* (H1). In the next action, Hanna claimed: *"the smaller the height is, the bigger the surface is"* (H2). The four girls discussed and checked H2 with their numerical data and sensed a conflict (C1). Liat proposed a new hypothesis (H3): *"Let's go the opposite way" when the container is smaller, the surface is bigger.* H2 and H3 were then confronted (conflict C2) in a very dynamic discussion during which numerical data were used as evidence, models were compared and clarification of the components of the problem situation was inquired. In a dialectic process between hypotheses, Osnat declared: *we were wrong in our formulation, the biggest the height, the smaller the surface* (rephrasing H2). Hanna, who felt that H2 and H3 were both supported by numerical data and that both could not be right, proposed to construct a table. But, nothing was done yet. Liat proposed a new hypothesis that bridges between H2 and H3, and that takes into consideration the local correctness of H2 and H3; she felt that the rate of change is not constant and that there is a "problematic segment" between 0.5 and 1 (values of the height). She made a gesture to indicate the "form of a parabola" (H4). Hanna made the same gesture and then formalized H4 by using the term "parabola".

Then they proposed again to construct a table. Miriam organized the construction of the table by varying the length of the base side. Then Miriam made another hypothesis: *"The smaller the side, the bigger the surface"* (H5). All the participants felt then that some data in the table do not confirm H5. They then tried to defend H5 by asking themselves whether some boxes were "wrong". Hanna explained that *"there is no wrong box, and if something doesn't succeed, it means that our idea is wrong"*. Osnat returned then to H1, and proposed to check again the computations in the table. Hanna defended H4 (the "parabola") by declaring that the function is not necessarily linear, meaning that all the computations fit the function. At this point, all the participants agreed that H4 is reasonable. They turned only then to the solution of the problem: constructing an algebraic model, drawing the graph with the graphical calculator, and proceeded to find the ideal dimensions as the minimum of the graph.

This description shows that many objects of previous actions became artifacts in further actions. For example, when hypothesizing H4 for the first time, the object is the hypothesis H4, and students use H2, H3 and numerical data as artifacts.

Another interesting phenomenon is that objects of actions gradually turned out to be immaterial: the objects of the actions are mainly hypotheses or refutations of hypotheses. The tendency to immateriality is also present in the fact that computer representatives are generally considered in an hypothetical mode, for example when the four girls use gestures to justify the "linearity" of the function-surface when the height is very big or very small, and that in the middle, the function is "curved to join the two lines" [sic]. These phenomena seem to indicate that computer transformations were internalized and that argumentation played an important role in this internalization.

The action of reporting (Phase 2) is not detailed here. We only stress an important phenomenon: the reflection expressed in the objects of reporting shows similarities and discrepancies with Phase 1: It seems to replicate its "inner flowchart" (see the parts marked in bold in Fig. 10). The group reported on two local hypotheses, but did not report on the precise hypotheses confronted in Phase 1. Some details of previous actions and regressions were deleted in the report. The changes left intact the general properties of the objects of the actions (for example that there was a conflict between two local hypotheses). The deletion of details and regressions left a document that referred to a substructure of the actual solution process, but not to its surface structure. Thus, the group constructed a new object, a new solution process, in which unimportant details were deleted and regressions were skipped. The remainder is the set of all the invariants of the action of reporting. We call this process, a process of *purification*.. Reporting was a commonly practiced during the whole year. Purification was ubiquitous in all reports.

### *Phase 3: The Synthesis*

The teacher opened the synthesis by asking students to *tell about the processes they went through*. At the beginning the discussion was focused on the ways they hypothesized the dimension of the ideal container. Such a request was commonplace in the classroom. The students were not asked for results, or even for simple descriptions of their previous actions. The synthesis is then the basis of new modifications of previous actions. One of the girls, Zippi, started to report, as shown in this first excerpt, the "cube hypothesis".

Zippi: *We took numbers with which we felt comfortable, to avoid fractions. To begin with, we did computations step by step, and then we did it more logically.*

She described the computations of the height and the surface area of the two models discussed in her group and declared:

Z(ippi):*Now, we saw that the second box was smaller than the first one, it's as if the surface is smaller, so we learned that the more the base is close to the height, perhaps it will be smaller.*
T: *Smaller? OK. Everybody understands Zippi's team's hypothesis? I want you to repeat her hypothesis.*
Zivit: The *closer the* $x$ *side is to the* $h$ *side, the smaller will be the result.*
T: *How does the ideal container according to Zippi, look like?*
*Rinat: A cube.*
*T:* *So, this is a hypothesis based on two examples. Is it correct or not?*
[Zippi remains silent; other students assert *"no, it is not the smallest"*]
T: (To Z)*So, was your hypothesis right or wrong?*
*Z:* *Not exactly. It was very close. It was the closest.*
T: *But as it is, the hypothesis was right or wrong?*
*Z:* [hesitates, then mumbles] *wrong.*
T: (to whole class),*OK. But it was a very beautiful hypothesis. Well, if somebody was asking you whether this hypothesis is reasonable or not, what could you say?*

This example shows that Zippi's verbal report and her interaction with the teacher is targeted to two levels. When Zippi solved jointly with her group Overseas in phase 1, she hypothesized mathematical hypotheses. Hypothesizing is an action, and hypotheses are the objects of hypothesizing. Following Zippi's report on the cube hypothesis, the teacher makes the students aware of the above distinction in her response: She first turns to the class, to note: *So, this is a hypothesis based on two examples* (relating to hypothesizing as an action). It is only then that she ostensibly asks Zippi: *Is it correct or not?* (relating to the cube hypothesis as an object). Zippi hesitates and her answer *Not exactly. It was very close* shows her perplexity about the correctness of what she did, because she relates to her hypothesis both as an action and as an object at the same time. The insisting question of the teacher about the correctness of the object-hypothesis leads Zippi to answer that the hypothesis (as an object) was wrong. But the teacher continues to stick to the above distinction and immediately praises her. Although she uses the term "hypothesis" she really means that Zippi's action of hypothesizing is beautiful. Thus, the teacher turns an action and an object constructed by a group to a shared artifact for the community of learners to encourage students to further action (verbal reporting/criticizing) on them. The action of hypothesizing becomes the object of the class discourse.

"Our" four girls took an active role in the continuation of the first example. They reported on their team work, starting by evaluating Zippi's "cube hypothesis":

Miriam: *First, I think that this hypothesis* (the cube hypothesis) *fits only a part of the function, but it's not the entire function. First, we also hypothesized this hypothesis, then another one.* [Miriam consults Osnat].

T: *Let's see which other hypothesis you raised.*

Osnat: *We were in a team and first, we saw we had two extreme examples, one with a small base.* [The students report on the examples 0.5*0.5* 9 and 3*3* 0.25]

T: *These are really extreme examples*

Osnat: *Then, we saw that the result, the surface, say when the side of the base is bigger, say, the result, the surface is smaller* [Equivalent to H1 or H3].

Liat: The *first one 18.25, and the second one 12.*

T: *So you based your hypothesis on the fact that it is smaller?*

Osnat: *No! We also thought that in fact, it's not a closed box . Now if the base will be bigger, we don't multiply it, but the height, we multiply by four. So perhaps, it's OK for the height to be small, and we multiply it by four...*

T: *Is this hypothesis clear? That is, you told us that the height has more weight..*

Hanna: *It's stronger because we multiply it by four.*

T (to the class):*Do you agree with the hypothesis that the height has more weight?*

Liat: *It's not exactly like that. We know that the two* [the base side and the height] *are linked.*

T (to the group): *Do you agree with this hypothesis after you solved the problem?*

Miriam: *I think that it can be like that for special cases, but generally, it's not like that.* [Mali, a girl from another group, intervenes].

Mali: *I think that I have an example. It's possible to see that for the first box, the height was 2.25. For the second box the height was 0.25, and in spite of that the first box had a smaller surface, in the first we got 10 and in the second 12.*

The discussion goes on; the girls report on the withdrawal from the local hypothesis.

Osnat: *Before that, we used the example that Mali gave, and suddenly, this changed the whole picture. Something contradicted and we tried to construct a table, that is to take a variable. And we took the side of the base to see what happens with the surface, and indeed we saw that it first decreases then it increases.*

The "cube hypothesis" never appeared in the discourse of our group in Phases 1 and 2. But, as a local hypothesis, which *"fits only part of the function"*, it reflects the "kind" of hypothesis (like H2 or H3) the group generated. Miriam appropriated here an artifact reported by Zippi that had the same property as those jointly constructed by her group. Osnat continues Miriam's verbal report to report on an additional hypothesis (with the help of Hanna and Liat who act here as if the group is one single person), and to characterize the examples on which the hypotheses were based as being extreme. The term "extreme" was not used by the group in phase 1 and 2. But, like for Miriam's appropriation, there is here a process initiated by Zippi's report, then the teacher's redistribution of the cube hypothesis as expressing valuable hypothesizing, its appropriation by Miriam followed by the integration of this hypothesis-object by other students to construct a new object. This object is more abstract in the sense that it is characterized by students by its properties (an hypothesis that fits a part of the function and based on extreme cases).

Osnat then reports on her hypothesis, H1 that is equivalent to H3 but was not accepted by her peers. Here also, the teacher leads a debate on the hypothesis as an object (*you told us, that the height has more weight.*) reinserting Osnat's hypothesis to the whole class. Liat rejects it (*It's not exactly like that. We know that the two are linked*) and Mali refutes this hypothesis by bringing one counter-example. Osnat's last intervention is a report on the end of the group solution process. This is an additional evidence for ***purification***: the reported process does contain any superfluous detail, and is not accurate as well. It is also compact, containing the sequels of many previous actions.

The teacher active use of reports on previous actions is normative in this class. Reflecting and reporting are acceptable mathematical activity where students can react, agree or refute, like Mali's counter-example. Verbal reports are not an appendix to the real mathematical activity, but an inherent part of it.

At this point of the synthesis, the teacher decided to descend to "real mathematical objects". The teacher devoted the last part of the synthesis to mathematical considerations of the solution. She initiated a lively discussion in which she asked: Which variable was chosen by the groups, the base side or the height? Do these two options lead to the same graph, the same function, and the same solution? If the two graphs are on the same coordinates, what is the meaning of the intersection point between them? If the function is considered out of its context (its domain being extended to all real numbers), how will the function graph look like in both cases? What is the rate of change for different parts of the function; or what happens where x tends to zero, etc. In this part the objects of the reflective processes are high level mathematical entities.

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### *Concluding remarks*

The Function Project provided students with an environment in which many transformations occurred. Actions, their objects and the artifacts that mediated their construction were modified during the phases of Overseas. The phases functioned on two levels. First, they helped iterating actions such as reporting on previous actions and reflecting on them from phase to phase. As such, the phases transformed the class in a community of practice whose resources (the available artifacts) were in constant change. This phenomenon was hardly visible within the Overseas activity but is reported in an ontogenic study elsewhere (Hershkowitz & Schwarz, 1995). However, the persistence children revealed during Phase 1 to hypothesize, and their openness in the synthesis indicates that hypothesizing, reporting and criticizing were more than actions carried out as a sign of obedience. At a second level, the phases within an activity (Overseas) helped students internalize external actions. For example, in Phase 1, the objects of actions served as artifacts for further actions and turned to be immaterial, often through the mediating role of argumentation. In Phase 2 and 3, reflecting in reporting expressed in *purification*, and synthesis helped to bestow meaning upon actions. In Overseas, the role of the computer seemed minor. However, we saw that students used the internalization of the manipulation of representatives (intensively practiced in earlier problem situations) to hypothesize the behavior of the function-surface.

### General conclusions on the two studies

This paper is an attempt to capitalize on two perspectives, cognitive and sociocultural, to characterize conceptual change in the case of functions. The contingency of two kinds of phenomena that occurred for $G_3$-students opens an opportunity to propose an explanatory frame for integrating the two perspectives. For example, $G_3$-students' proficiency to explain (as it appears in their written answers to the questionnaire) seems an expression of the actions in which students were involved during the phases of activities such as Overseas. $G_3$-students gained practice to report, criticize and reflect on their own and others' solution processes. The use of prototypes in a flexible way as "levers", to judge new examples is echoed in the fact that the partiality and the ambiguity of function displays (representatives) supported dialectical processes to converge from local to global considerations. In Overseas, the surface-function emerged as the construction of a shared meaning embracing more and more data. The function-surface that seemed "partially linear" to the members of the small group "became non-linear" as a result of discursive actions to consider another perspective for agreeing, checking, or contrasting views. Therefore, reasoning about functions often referred to linear or quadratic functions but dialectic processes resolved the exact nature of the properties of the function. The intensive

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involvement with problem-situations also led students to consider the soundness of results regarding the situation they referred to. We evaluate that the positive results $G_3$-students attained, are also linked to practices already internalized in Overseas. During the year, students' manipulations included the passage to a new representation and the flexible use of partial displays which were conciliated to make up judgments. The efforts we put in integrating cognitive and sociocultural findings in the same analysis are at their beginning, but we believe that the application of the Activity Theory will allow more bridges to be constructed.

## REFERENCES

Dreyfus, T., Hershkowitz, R. & Bruckheimer, M. (1987). Processes in the transition from syllabus to curriculum. Zentralblatt fur Didaktik der Mathematik, 19(1), 19-25.

Hershkowitz, R. (1989). Visualization in geometry: two sides of the coin. Focus on learning problems in mathematics. Vol. 11(1), (pp. 61-76).

Hershkowitz, R. & Schwarz, B. B. (1995). Reflective processes in a technology-based mathematics classroom. Paper presented at the AERA, San Francisco.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 515-556). Reston, VA: National Council of Teachers of mathematics.

Karplus, R. (1979): Continuous functions: student's viewpoints. European Journal of Science Education, 1, 397-415.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. American Educational Research Journal, 27, 29-63.

Leonte'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed.), The concept of activity in Soviet psychology (pp. 37-71). Armonk, NY: Sharpe.

Markovits, Z. (1982). Understanding of the concept of function among Grade 9 students. Unpublished Master Dissertation, Weizmann Institute, Rehovot, Israel.

Rosch, E. & Mervis, C. B. (1975). Family resemblance: Studies in the internal structure of categories. Cognitive Psychology, 7, 573-605.

Schwarz, B. B. & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective of functions. Educational Studies in Mathematics, 259-291.

Vinner, S. (1983). Concept definition, concept image, and the notion of function. International Journal of Mathematical Education in Science and Technology, 14, 295-305.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. Journal for Research in Mathematics Education, 20(4), 356-366.

Vinner, S., & Hershkowitz, R. (1983). On concept formation in geometry. Zentralblatt fur Didaktik der Mathematik, 15, 20-25.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458-477.

# Emergence of new schemes for solving algebra word problems: The impact of technology and the function approach

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*This article suggests that thinking about technology and functions as the foundation of post arithmetic curriculum changes roles, reasons, models of problem solvers and foci of research. The first part suggests principal changes of thinking about functions and algebra as a result of the use of technology. The second part deals with strategies of research and methods of analysis of longitudinal studies on learning the topic of functions and algebra. The third part suggests that viewing algebra as a study of functions aided by technology, shakes common models of problem solvers, particularly those of weak students. The forth part includes a systematic analysis of types of algebra problems according to situation models and algebraic technique, and suggests a rethinking about the complexity of motion algebra word problems modeled by functions.*

## Solving Word problems in Algebra: An Overview on Modeling, Functions and Technology

Modeling can be viewed as thinking about one thing in terms of simpler artificial things. Thus, the intermediate stages that precede the symbolic formulation of a model should help simplify the text, peel off details, and describe it in what is referred to as a situation mode. The most conventional setting for algebraic representation of authentic situations is "word problems" or "problems in context." Here, students are presented with a linguistic description of a situation -- a word problem -- and are asked to create an algebraic model and solve it by manipulating symbols. In elementary algebra courses, expectations have always been that problem solvers would assign symbols to variables such as distance or speed, arrange them in an algebraic relation, solve the equation and answer the questions posed. Less traditional aspects of problem solving in algebra are presented by Hall (Hall et. al, 1989). Hall sudied problem solvers who were presented with algebraic story problems about motion and work. The main findings suggest that solvers of word problems in algebra devote a substantial portion of their work to re-representation of the problem at the situation level. Hall argues that "...the situation context of an algebra sory problem and, in particular, the correspondence between situation relations and quantitative constraints should be legitimate objects of teaching in the algebra curriculum." (p 276) I think, like Hall, that a major agenda of algebra teaching should be equipping learners with tools for mathematizing the perception of the situation context, and I argue that placing the function as a central object of the learning could support this evolution of mathematization.

Because of its power to bring the real world into the classroom, technology has been seen to play a central role in empowering students to create and understand mathematical models. Technology can serve as a bridge between the science lab and traditional algebra and calculus problems. Technology can also support the use of new ways of symbolization and in its most spread mode it supports the graphical and numerical representations of symbolic expressions. Using these capabilities, technology offers various options for solving algebraic word problems. It can support meaningful modeling of traditional problems without any use of algebra (e.g. The *Trips* software, Clements and Nemirovsky 1995, offers an environment for posing and solving motion problems using simulations, graphs and Logo). Spreadsheets, graphic software and symbolic manipulators can even provide symbolic models for fitting given data, and manipulating equations created with these models (Heid 1995, Fey 1989). Another attempt to support mathematization with software is made by interactively creating semantic schemes of quantities while modeling and graphing algebra word problems (Schwartz 1991, Cuoco 1992). In all the above mentioned approaches, the function is a central concept of the model that supports the understanding of problems. It is not obvious, however, how the function model effects the evolution and understanding of symbolic models and the manipulation of symbolic relations. Our studies attempt to explore the role of the function, both its symbolic structures and its qualitative aspects, in solving algebra word problems in a graphic technology environment.

## 1. The research platform: a curriculum that changes scope and sequence

The preparation of an appropriate and interesting setting for the investigation of students' actions in this new territory of algebra, namely, modeling and solving, is a complicated task (Brown 1992). Such a study could only be interesting and meaningful if both the mathematical content and tasks, and the actions of learning and teaching in the classroom, make the classroom an appropriate learning environment. The typical traditional classroom cannot usually form such an environment. The appropriate setting for such a study is thus hard to create because it requires not only new books and mediated activities but also a reformulating of classroom discourse. Representations, tools, software, and activities are necessary but insufficient for creating a community of learners motivated to communicate their mathematical ideas. For that to happen, the roles of both teacher and student must change. Such a change requires a professional who is willing to explore this kind of innovation over a long period of time. The platform we prepared for the study includes a full sequence of algebra course supported by innovative software tools, especially designed materials for students and teachers, and workshops and materials for professional development.

### 1.1 Sequence: the organization and flow of the curriculum

The experimental curriculum -- "*Visual Mathematics*" (CET 1995) -- is based on the idea that, for pedagogical reasons, the function is the appropriate fundamental object of secondary school mathematics, and focusing on the concept of function allows the organization of algebra curriculum around major ideas rather than technical manipulations (Yerushalmy, Schwartz 1993). We are exploring these basic principles as part of an algebra curriculum adopted by several schools (grades 7 to 9) in Israel. The curricular sequence of this algebra course is organized in three major phases, as follows: (1) building the concept of function, (2) comparing functions, and (3) exploring the invariance (families) of functions. In each phase, functions are represented in four different forms: numbers, graphs, symbols, and words. Troughout the curriculum, modeling, manipulating, inferring and communicating (as defined in Schwartz, 1995) are the major building blocks of the curricular activity. In order to understand the first phase, one could imagine a tetrahedron, whose vertices are the different constructs and representations: symbolic language, numerical language, graphical language and natural language. Ultimately, our aim is that students will be able to use all these representations with some agility. Traditionally, instructions are formulated so that students proceed from symbols to numbers and then (sometimes) to graphs. Thus, students must master algebraic rules and manipulations before they can use mathematics for modeling the world. We believe that this is one of the reasons why so little time is devoted to important mathematical inquiry in the secondary school curriculum. This way of teaching also implies that students spend most of their learning time manipulating symbols without even being able to connect what they are doing to important mathematical ideas. In our suggested new formulation, it is possible, with the aid of technology, to begin with any of the representations of the function and then to proceed to any other representation. Throughout the second and third phases, the emphasis shifts towards symbolic manipulations of expressions and relations. Words and graphs are used to describe the effect of the symbolic manipulations, and symbolic language is used to express and explore the nature of the function in greater delicacy than can be done otherwise.

### 1.2 The scope of the technology and function approach

The use of technology and its graphic and numeric features, requires and allows not just a change in the sequence of learning but rather a deep change in the scope of traditional actions and objects. Following is a partial list of such actions and objects that we believe are central to the changes observed in the problem solving activities that will be described later on.

**Sketching from a story by observing rates of change:** The basis for the function's conception and presentation is the sketch of events and processes of temporal phenomena. The sketches are built and analyzed using mathematical

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**properties of variation.** The learning of function begins with reformulation of narratives using two lexical sets: verbal and iconic. The two sets are parallel representations of the characteristics of any process that can be described by a function of a single variable. A software environment (The *Algebra Sketchbook*, Yerushalmy, Shternberg 1993) provides an opportunity for manipulating visual objects and using a verbal set which is simpler and more abstract than natural language descriptions. The use of these lexical sets offers a path toward modeling that does not require the use of algebraic symbols, traditionally the only first route of employing mathematical notations.

**Thinking continuously and directionally:** Modeling with functions of real numbers immediately suggests the use of a Cartesian system which is a directional system. Thus, any quantity involved in the model must be identified by size and direction. This is not usually the case while expressing processes using algebra. The use of technology offers ways for dealing with continuous processes rather than with descrete points which are often the basis for modeling and symbolizing in algebra. Using special options of the software, the continuous graph which is usually treated as constructed from points, turns to be a tangible object which can be manipulated by itself.

**Comparison of functions as the conceptual model of equations:** Traditionally, solving equations is an analytic, non contextual task that has been often considered to be subject to rote learning. Technology which provides numerical solutions mainly through the use of graphical analysis, allows us to split the traditional meaning of "solving an equation" into two parts: manipulating equations and seeing solutions, which are no longer mutually dependent. Once symbolic manipulations are disconnected from the process of reaching a solution, major concerns are attributed to the mutual relations between two processes.

**Letters as arguments, variables and unknowns:** The natural choice is to represent by a letter the independent quantity, the function's argument. Thus, while in traditional algebra the letter represents the unknown, and the symbolic description (or the equation) is built in order to find the unknown, in the function approach, the first need for symbolic generalization is embedded in the description of the independent quantity (Herscovics 1989 ) Consequently, when more than a single letter is involved, the expression is precept as a multi-variables function and not as a parametric expression nor as a symbolic description of objects (Kuchemann 1978 ) Often, modeling of temporal phenomena requires the use of letters in more than a single mode. Letters may represent static objects such as events (e.g. the coordinates of a point) or a varied geometric measure of an object (such as a letter representing the length of a segment in a coordinate system..)

**"Numbers" as functions:** In the function world, numbers, like letters, have reference. Therefore, any number should be interpreted either by the geometric property derived from the coordinate system (e.g. numbers indicate points' values, lines or planes) or within the expression in which it appears ( e.g. $x + 6$, can be thought of as a sum of two functions, as a single function or as a translation of each point by a quantity of 6).

## 2 Development of problem solving strategies throughout the three years

The construction of mathematical tools and habits that make the function, conceptually and symbolically, the habit of mind and the natural choice for problem solving is a long learning process. Providing a set of tools (concepts, representations, technology etc.) which includes more than just the necessary procedures, was aimed at allowing various learners a choice of tools according to their needs, personal preferences, ability etc. Thus, research of the opportunities and choices used requires a long term investigation. Having a full curricular sequence of algebra centered around these concepts allowed us to plan longitudinal studies and follow the evolution of algebraic knowledge during three years. The methods of study and the aspects of the analysis of the long term investigation are described below.

### 2.1 Research methods

During three school years (grades 7 to 9), we interviewed the same 12 pairs of students twice a year. The students, all participating in the "*Visual Mathematics*" program in a compulsory algebra course, came from two schools and four different teachers. In the interviews, the pairs were presented with written word problems and were allowed to solve them in any way they chose. They could use paper and pencil, calculators, and graphic software. The tasks used ranged from standard algebra problems that were repeated along the five interviews, to non-standard problems such as problems involving only graphical data or problems in which the students were asked to describe and model rather than solve. We chose relatively simple situation problems as the repeated tasks because this type of questions does not determine which mathematical tools the interviewee should use , and thus allowed us to examine the unconstrained choice of the mathematics tool, representation or strategy for each question during the three years. The other problems were more complicated tasks which had not been studied or presented in the same form in class. Each interview included two or three problems of both kinds and lasted between 60 to 90 minutes. **Pair interview** was chosen as the method of interview for two reasons: Working with the software in pairs was the usual method of work in the math classes , and thus it provided the interviewees with a natural learning environment and allowed the interviewer to watch their actions in a natural setting. Secondly, the pair interview is assumed to be a more natural setting than an individual "thinking aloud" process (Scheonfeld

1985). **The sample** was drawn from four classrooms taught by different teachers, in two schools with different populations and different socioeconomic conditions.. Three pairs were chosen from each class while observing their class at the beginning of the 7th grade. One pair consisted of students who understood the taught topics, tended to explore new avenues, cooperated with their classmates and teacher in developing new ideas, and were productive in completing tasks. We usually refer to these students as the **"Able math students"**. Another pair consisted of students who in general seemed less happy in the classroom. Those were students who needed extra explanations, more demonstrations and directions in performing computer tasks, and often, during the whole class discussion, presented a clear dissatisfaction. Such students are Normally called weak students, and here we refer to them as the **"Less able pair".** The remaining pairs were **"Average ability"** pairs, consisting of students who for the large part succeeded in performing the tasks but did not develop often new ideas. The criteria for choosing the interviewees were: students participation in class and small group discussions, verbal ability and tendency, cooperation with the teacher and the classmates, attitude towards work in the computer lab and tendency to cooperate both with their partner and the interviewer.

## 2.2 Analysis Methods

Many aspects are involved in such problem solving activities, but the current analysis concentrates on three cognitive aspects which are most liable to capture the effects of the scope and sequence described above. These aspects are: 1) Phases, actions (with and without technology) and flow of the problem solving process -- all grouped together under Attempts, 2) the representations, perception of patterns, and the methods and models used for solving equations -- referred to as Tools, and 3) the characteristics of the relation between the situation story and the solution.

**Attempts:** The typical profile of algebra problem solvers identifies two phases of solution: problem comprehension and equation solving (Chaiklin 1989). The problem comprehension model involves (a) reading the problem, (b) forming a mental representation that interprets the information given in the problem into objects with associate properties, (c) organizing the relations among those objects, and (d) representing the relations by equations. This last phase helps to shape the mental representations required for solution. The equation solving phase requires strategic knowledge in order to manipulate the equation. Repeated findings about expert and novice problem solvers point to the ability and tendency of experts to integrate problem comprehension and solution manipulations (Schoenfeld 1987).But, even when both experts and novices succeed in solving and manipulating equations, they differ in their ability to provide meaningful interpretations of the equations which can help them comprehend the problem. In our study, we expected to find several differences in the behavior of novices compared to the one described above. As throughout the curriculum, an emphasize was made on describing rather than solving, the behavior of novices identified by Schoenfeld (1985) and others, of a quick linear process of reading and planning followed by a longer implementing phase should have been less frequently observed, and a mix between the phases has been expected. Since the software with both its graphical and numerical tools was part of the problem solving environment, and there was an option of reasoning about the result without actually arriving at the result by algebraic manipulations, the interpretation of the result did not depend any more on the ability to manipulate equations. Besides being an aid in the solution itself, the software was expected to support the experimentation, planning and argumentation weaved within the attempts at solution.

**Tools:** The use of multiple representations to solve algebra problems is a major issue in the functions and technology approach, one that had been previously discussed at length in many works (Romberg, Fennema, Carpenter 1993, Kaput 1989 ). Lately it had been discussed and suggested that personal tendency and ability to use specific representations is hard to identify, and cannot be used as a means of explaining differences between learners. In particular, with the use of parallel linked representation software, the actions taken involve a blend of the different representations. Therefor, in the current analysis we do not consider so much the type of representation but rather the strategy of patterns formation in any of the representations. The strategies include anecdotal representations of data, data organized by co-variation and analysis of variation. In an algebra course, one does not usually look for a pattern but rather manipulates symbols and numbers. Algebraic expressions that describe functions or rules are usually formed using co-variation, while analysis methods and model formation often make use of variation (Confrey, Smith 1994)

| | *Numbers* | *Graphs* | *Symbolic* |
|---|---|---|---|
| Anecdotal | 2,16, | | $x, y, a$ |
| Co-variation | [illegible] | | $f(x) = ax + b$ |
| Variation | [illegible] | | $\frac{\Delta y}{\Delta x} = a$ |

The other set of tools involves forming and solving equations. Obviously, we expect less than the usual intensive use of procedures. The analysis concentrates on the models of solutions rather than on techniques, manipulation ability or technical errors. Regarding the scope of the curriculum, we expect a shift from models of equation held by arithmetic students (Kaput 1989) towards a massive use of the function's comparison model. The use of the comparison model ($f(x)=g(x)$) is especially central and critical to our conjecture that the situation model, if consisting of processes described as functions, serves as the mathematical structure which is attached both to the story and to the analytic equation. The use of this model should encourage the tendency to describe rather than solve, since it requires thinking about processes and relations rather than on the unknown (Chazan 1993). It is also expected that attempts of solution will be done in any of the representations, according to the stage of learning, ability and personal preferences.

**The situation context and the solution**. Traditionally, modeling is assumed to occur at two phases, as described by Davis and Hersh (1981): the real physical and the ideal mathematical. Janvier (1996) regards the process of modeling "to involve a formulation phase that is completed by a validation phase. In the formulation phase: the establishment of relationships between the variables involved. Those relationships originate from observations or measurements, or simply a clever guess. Traditionally, the relationship was expressed as a formula expressing one variable in terms of the others. Recently it has become common to consider also as a model the graph or a table of numbers produced as a result of a computer simulation". These descriptions concentrate on the transition between the two phases, while other works suggests the existence of another phase between the two phases: the situation model (Nathan, Kintch, Young 1992). The identification of existence and type of connection between the situation and mathematics is the major role of this part of the analysis. Using the situation to improve the solution was found by Hall (*ibid.*) to appear in four modes: 1) as a first exploration, 2) as a solution strategy when the student is on track, 3) as evidence when the student has already found the solution, 4) as a recovery strategy when the student suspects that he is off track. On the other hand, Chaiklin (1989) reports that the reality of the situation does not affect those students who form a structural model of the problem. They will solve algebra problems in the same method and rate of success whether the problems are meaningful or not. Undoubtfully, the sequence of learning taken here should have an impact on this problem solving behavior. The use of software for modeling is also expected (as partially described in Yerushalmy 1997) to have an impact on the formation of a situation model that links the situation to the analytic model. Noble & Nemirovsky (in press) look at a tighter connection between the situation and the mathematics - they call it Fusion. The fusion between the phases seems to be an appropriate way to describe modeling while working with dynamic software, "...telling the story of a graph can become an experience of imagining moving along the graph oneself and of seeing the fusion of the lines, points, etc. of the graph with the events the graph represents." (p.7). The fusion we expect here should take a slightly different form: a fusion between numerical patterns or symbolic expressions and equations, and the situation story.

## 3. Comparative observation: What is weak in the "weak" students performance?

The following is a description of expected behavior of weak problem solvers by Cardelle-Elawar (1995): "Low performing students commonly exhibit counter-productive behaviors in approaching mathematics problems. For example, (a) they read rapidly at the expense of full comprehension of the problem; (b) they do not reorganize information; (c) they do not recognize that there might be more than one right way to solve the problem (d) they are uncertain about how to calculate and verify solution, (e) they give up easily when they do not know how to approach the problem." (p.81) Throughout our long term study, students who take the traditional algebra and functions course (where functions are taught separately from algebraic concepts and manipulations, and without technology) were also interviewed in pairs. In my presentation we will watch short segments of a single videotape (Yerushalmy, Sion 1996) which will help us identify weaknesses and strengths in two pairs solving the same simple algebra problem. One pair consists of a boy and a girl of the 10th grade, with past and present success in math. They had completed algebra I course and are solving word problems both in algebra courses and in physics. They volunteered to take the interview although they are not studying the "*Visual Mathematics*" experimental curriculum. The other pair, two 9th graders, is one of the pairs belonging to the category of the less able pairs. The interview was taken during their third year in the longitudinal study. The video shows 10 minutes out of half an hour interview with each pair. Watching the last few minutes of each interview one has to come to the conclusion that both pairs were able to solve the problem successfully and talk about the semantics - the meaning of the solution in terms of the situation. The rest of the interview, however, is totally different. Watching these two pairs suggests the emergence of new models of novice problem solvers when working with technology in algebra.

The first pair, S&A, started by questions regarding the meaning of the task. Once they understood the language, they plugged in symbols and started to arrange the symbols into expressions and equation to get a numerical solution. When they felt that they chose the symbols correctly, they put the situation aside and manipulated the symbols. Twice they stopped after a few minutes of manipulations, asking, "What is all that good for? Why are we doing that?" They decided to start all over again. Time after time they start over, not providing any reason regarding the situation, concentrating on symbols, alone and attempting to change their data organization method. Finally, once they had a well formed two variables equation, they realized that since one variable can be canceled in both sides, they could drop it a priory and get an equation of a single variable -- which they can then solve. Only after this relatively long work they went back to the semantics - suggesting that they can prove something about it, realizing that actually they are not supposed to find a single answer but a range of possibilities (which they called an inequality), and then were ready, upon request, to explain the meaning of the solution. During all that time you will see a very happy pair of kids, working on a simple problem that should have been known to them, and failing to propose a decent solution but not getting frustrated. They are sure that they are, or shortly will be, on the right track, they are confident in their ability, they are not discouraged by the fact that they have to start over even though they don't have any new meaningful idea how to start over ,-- they are confident that they would deal with it successfully -- probably because of their history of success -- and are sure that the symbolic tools and manipulations they fluently use will bring them, as always, to a safe shore.

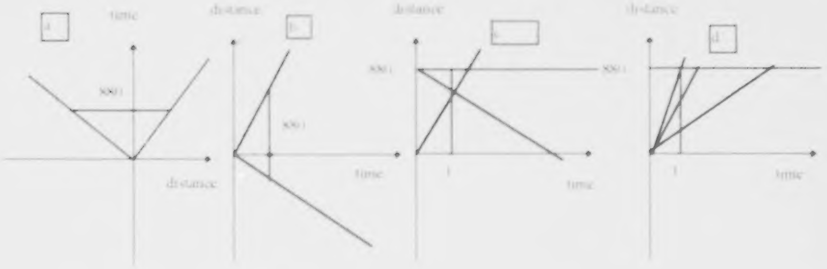
The other pair, Y&G, presented a promising start: they had read the problem, understood what is required, modeled both processes symbolically and compared them to formulate the required equation. While doing so they repeatedly talked about the semantics, the need to do and use algebra the way they did in order to represent the given situation. If they had been what we call "strong students", with a reputation of successful math students, they would have probably finished the solution within another minute. Things were not that smooth, however, and it took them much longer. When they started to manipulate the equation, they made a mistake and stopped working on the equation. The atmosphere was very dense, and they were stressed and quiet. According to the model of weak math students they would have given up at this stage. But they did not . Y suggested that if they sketch graphs, they will be able to see something about the modeled problem. When that was completed but did not provide any new information about the solution, they asked to use the graphic software. Using evaluation of rates and variations they could make sure that their expressions are correct and as they already had the appropriate paper and pencil sketches, when the graphs appeared on the screen they could recognize immediately what they were looking for in the equation: "Here it is! it intersects!". From there on they described and explained the situation and the solution using a mixed language of screen information and the story. They looked happy, they had a lot to talk about, and it all made a lot of sense. If one watches only the first and last two minutes, one gets the impression that here are successful algebra students. Watching the middle segment, however, leaves the impression of students who are nervous with symbolic manipulations, feeling stuck with mathematics, even ready perhaps to admit another failure.

The question arises what kind of algebra are the two pairs performing. Should we characterize the performances as weak and strong? What are the properties of this approach that brought the students to their present level of performance? What would have happened if G&Y had a symbolic manipulator? Are the problems we usually see as algebra problems really algebra problems?

## 4. How simple is a simple algebra problem?

Our further investigation of attempts to solve algebra problems lead us to acknowledge the fact that the technology and functions approach might even lead to a substantial change in the way we see traditional tasks. We will use a few examples to explore whether situation models that are constructed by functions, can be at the same time, close to the authentic story and suggestive to the analytic model. The first example presents four ways to think about the same situation and the algebra done as a result of one's choice. The second presents three "algebraic equivalent" problems that consideration of their function's models suggests major differences that are otherwise invisible.

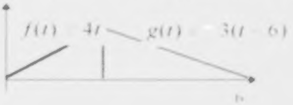
The first example is drawn from Hall's (*ibid.*) work. "Two trains leave the same station at the same time. They travel in opposite directions. One train travels 60 km/h and the other 100 km/h. In how many hours they will be 880 kms apart?" (p.225) Hall's students charted the situation as a single dimension chart of distances and speeds, but some did draw a diagram like the one in figure a. While this graph appropriately charts the situation, figure (b) is probably a more natural way to describe such temporal phenomena. It has time as its independent variable and the speed can be read as the slope of the function. But the equation $100t + 60t = 880$ describes a sum of two segments' lengths with $t$ as the unknown. Neither its left nor its right side describe any function that appears in the figure. What then is the story behind figure (c)? Geometrically, it is an equivalent graph (think about a vertical translation of one function). It allows to read the solution numerically from the graphs of $f(t) = 100t, g(t) = 880 - 60t$, and it describes two trains that start to move from two stations situated 880 kms apart. Figure (d) models the sum of distances as an addition of functions: $f(t) = 100t, g(t) = 60t, h(t) = 100t + 60t, k(t) = 880$, solve $h(t) = k(t)$, and it ignores the actual directions of the motions. The equation is identical, however, to the one described by figure (b), except that it presents a sum of functions and a constant function rather than a sum of segments' lengths. Is there a specific model that would be best for solving the problem posed in this example?



The second example deals with three versions of a story about a round trip of a biker whose speed on one way is different from the speed on the way back.

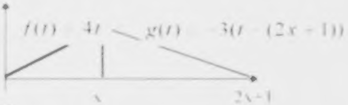
| | *time* | *speed* | *distance* |
|---|---|---|---|
| out | x | 4 | 4x |
| back | 6-x | 3 | 3(6-x) |

$4x = 3(6 - x)$



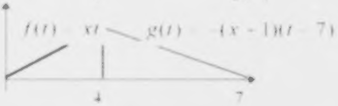
| | *time* | *speed* | *distance* |
|---|---|---|---|
| out | x | 4 | 4x |
| back | x+1 | 3 | 3(x+1) |

$4x = 3(x + 1)$



| | *time* | *speed* | *distance* |
|---|---|---|---|
| out | 4 | x | 4x |
| back | 3 | x+1 | 3(x+1) |

$4x = 3(x + 1)$



Using an organizing table for the given data (the left column of the figure) supports the formulation of three algebraically equivalent equations. But the functions' models (the right part of the figure) tell about three different types of algebra problems. The first situation can be solved without any algebra, since the functions can be explicitly described as two known processes and the intersection produces the result. The second situation cannot be graphed but rather sketched. It requires assuming an unknown, and describing the second motion as its function. The figure therefor, describes a model of the situation and not "the" situation and its solution cannot be read from the figure. The third describes functions of time as the independent quantity and speed as a parameter and is even more complicated algebraically if viewed as a comparison of two processes. Do we have to blame the function approach for these complexities, or is it because mathematical modeling with algebra in a non rote way is a more complicated task than we used to think about?

We started to watch for these differences between problems in a more systematic way and analyzed an exhaustive list of traditional algebra word problems when treated with graphic technology and the function's approach. This analysis will help us to reconsider problems such as "what makes an algebra word problem simple or complex?," what is a good algebra problem," or "what is a worthwhile algebra problem?"

## References

Brown, A. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. The Journal of the Learning Sciences, (2),141-178.

Cardelle-Elawar, M. (1995) Effects of metacognitive instruction in low echievers in mathematics problems. Teaching & Tecahers Educaiton, 11, 81-95.

Center for Educational Technology. (1995) "Visual Mathematics: Algebra". Written materials and software. Published by the author, (In Hebrew). Tel-Aviv.

Chaiklin, S. (1989) Cognitive studies of algebra problem solving and learning. In S. Wagner, C. Kieran (Eds.) Research Issues in the Learning and Teaching of Algebra. (pp. 93-114). Hillsdale, NJ: Erlbaum.

Clements, D. H. Nemirovsky, R. (1995) Trips, Computer software. Dale Seymore.

Confrey, J. Smith, E. (1994) Exponential functions, rates of change and the multiplicative unit. Educational Studies in Mathematics. (26), 135-164.

Cuoco, A. (1992) Actions to process: Constructing functions from algebra word problems. Report 92-1 Center for Learning Teaching and Technology, Education Development Center, Newton, MA.

Davis, P. J., Hersh, R. (1981) The Mathematical Experience. Pinguin Books.

Fey, J. T. (1989) School algebra for the year 2000. In S. Wagner, C. Kieran (Eds.) Research Issues in the Learning and Teaching of Algebra. (pp. 199-213). Hillsdale, NJ: Erlbaum.

Heid, K. (1995) Algebra in a technological world, Curriculum and Evaluation Standards for School Mathematics. National Council of Teachers of Mathematics (NCTM) Reston, VA.

Herscovics, N. (1989) Cognitive obstacles encountered in the learning of algebra. In S. Wagner, C. Kieran (Eds.) Research Issues in the Learning and Teaching of Algebra. (pp. 60-86). Hillsdale, NJ: Erlbaum.

Janvier, C. (1996) Modeling and the initiation into algebra. N. Bednarz, C. Kieran, L. Lee (Eds.). Approaches to Algebra: Perspectives for Research and Teaching.(pp. 225-236). Dortrecht, The Netherlands: Kluwer.

Kaput J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner, C. Kieran (Eds.) Research Issues in the Learning and Teaching of Algebra. (pp. 167-194). Hillsdale, NJ: Erlbaum.

Kuchemann, D. (1981) Algebra. In K.M. Hart (Ed.) Children Understanding Mathematics. 11-166 (pp. 102-119). London: John Murray.

Hall, R., Kibler, D., Wenger, E. Truxsaw, C. (1989). Exploring the episodic structure of Algebra story problem solving". Cognition and Instruction, 6(3), 223-283.

Nathan, M.J. & Kintch, W., Young, E. (1992) A theory of algebra word problem comprehension and its implications for the design of learning environments. Cognition and Instruction, 9 (4), 329-389.

Noble, T. Nemirovsky, R. (In press) Graphs that go backwards. Paper in progress.

Romberg, T. A., Fennema E., Carpenter T. P. (1993) Integrating Research on the Graphical Representation of Functions.(Eds.) Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (1985) Mathematical Problem Solving. Academic Press.

Schoenfeld, A. H. (1987) What's all the fuss about metacognition? in A. H. Schoenfeld (ed.), Cognitive Science and Mathematics Education. (pp. 189-216). Hillsdale, NJ: Erlbaum.

Schwartz, J.L. (1995) Assessing mathematical understanding and skills effectively. An Interim Report of the Harvard Group Balanced Assessment in Mathematics Project, September, Educational Technology Center, Harvard Graduate School of Education.

Schwartz, J.L. (1991) The Algebraic Proposer, Computer Software, True Basic Inc.,VT.

Yerushalmy, M. (1997). Mathematizing qualitative verbal descriptions of situations. A language to support modeling. Cognition and Instruction. 15, 207-264.

Yerushalmy, M. Sion, M. (1996) Thinking Algebra. Video tape.(In Hebrew and English)

Yerushalmy, M., Shternberg, B. (1993). The Algebra Sketchbook (computer software). Tel-Aviv, Israel. Center for Educational Technology.

Yerushalmy, M. ,Schwartz, J.L. (1993) Seizing the opportunity to make algebra mathematically and pedagogically interesting In : T.A. Romberg, E. Fennema, and T. Carpenter (Eds.) Integrating Research on Graphical Representations of Functions, Erlbaum Inc. NJ.

**Research Forum**

## RESEARCH ON MATHEMATICAL PROOF

| | |
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| *Coordinator* | Carolyn Maher |
| *Presentation* | Approaching geometry theorems in contexts: from history and epistemology to cognition<br>by Maria Alessandra Mariotti, Maria G. Bartolini Bussi, Paolo Boero and Rosella Garuti |
| *Reactors* | Guershon Harel<br>Michael de Villiers |

# APPROACHING GEOMETRY THEOREMS IN CONTEXTS: FROM HISTORY AND EPISTEMOLOGY TO COGNITION

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*This paper presents the common theoretical framework and the main findings of three long-term research studies carried out over the last five years by teams in Genoa, Modena and Pisa. These studies have involved students of different age groups (from grade 5 to grade 10) and different fields of experience, namely the representation of the visible world by means of geometrical perspective; sunshadows; geometrical constructions in the Cabri environment. The paper introduces an historico - epistemological analysis of mathematical theorems as unities of statement, proof and theory, where the conditional form of the statement plays a major role. To approach geometry theorems in this sense, the features of the field of experience and of the teacher's role in classroom interaction are analysed. The functions of dynamic exploration in the generation of the conditional form of theorems and the proving process are discussed.*

## 1. Introduction

This paper is based on research work regarding the approach to geometry theorems in schools carried out over the last five years by teams in Genoa, Modena and Pisa. These studies have involved students of different age groups (from grade 5 to grade 10) and different thematic contexts. Although the specific goals of these projects differed to some extent, they did share some common features such as general goals, research methodology, epistemological analysis and cultural, cognitive and educational hypotheses. A common framework is emerging as a result of a longstanding dialectic discussion dating back to the design of our teaching experiments: this framework has brought to light some of the deep yet implicit common motives and theoretical perspectives of our independent research designs. This paper provides a unified framework and an original survey of the research studies and their findings, which have been partially reported in other papers (Bartolini Bussi, 1996; Bartolini Bussi & Bergamini, 1996; Costa et al., 1994; Boero et al., 1996; Garuti et al., 1996; Mariotti, 1995b; 1996). We will focus on the following general points: the function of the different contexts in approaching geometry theorems; the role of the teacher in classroom interaction; and the idea of theorems as unities of statement, proof and theory. In this framework we shall analyse the function of dynamic exploration for approaching geometry theorems in those contexts at different school levels.

Our studies take into account the following issues in current research into the school approach to theorems: the present-day value of proof in mathematics and mathematics education (Hanna, 1996), even for very young pupils (Maher, 1995); the social dimensions of the approach to proof (Balacheff, 1991) and the distinction between argumentative reasoning and deductive reasoning (Balacheff, 1988; Duval 1991); the classification of student proof schemes (Harel 1996) and the relevance of 'transformational reasoning' in the production of statements and the construction of proofs (Simon, 1996); the study of the potentialities of geometrical software (Goldenberg & Cuoco 1996; Laborde, 1992; 1993). Inspired by the seminal work of Balacheff (1988) and new studies (e.g. De Villiers 1991; Hanna & Jahnke 1993) on the pragmatic of proof, we focused on the link between epistemological, cognitive and didactic analysis.

## 2. Theoretical Framework

The general theoretical framework of our research studies is based on the construct of 'field of experience' and the construct of 'mathematical discussion'.

As reported in Boero et al. (1995), a field of experience can be metaphorically defined as a system of three evolutive components (external context, the student's internal context and the teacher's internal context) referred to a sector of human culture which the teacher and students can recognize and consider as unitary and homogeneous. Classroom activities within any field of experience can have different goals: in this paper we shall limit ourselves to those related to approaching geometry theorems. In this perspective, the features of a field of experience that are meaningful for us can be described as follows:

- the presence of 'concrete' and semantically pregnant referents (external context) for performing concrete actions that allow the internalisation of the visual field where dynamic mental experiments are carried out; this feature is consistent with Vygotskij's general theory on mental processes and with specific findings on the function of dynamic processes both in the production of conjectures and in the construction of proofs (see Polya's idea of variational strategies, as well as the recent consideration of 'transformational reasoning' by Simon, 1996);
- the presence of semiotic mediation tools (including excerpts from historical sources, documents, meaningful linguistic expressions), chosen by the teacher from the cultural heritage with the aim of introducing the mathematical idea of theorem;
- the construction of an evolving student internal context, rooted in the dynamic exploration, where different processes such as conjecturing, arguing, proving and systematizing proofs as formal deduction are given sense and value (see 2.1.).

These points are consistent with general ideas about the production of geometry statements and the construction of proofs relying on the one hand on 'reified' (Sfard, 1991) pieces of knowledge produced by the historical evolution of mathematics and, on the other, on figural (Fischbein, 1993) referents, which may be either static or dynamic (Dreyfus, 1991).

As concerns mathematical discussion, we refer to the metaphorical definition given by Bartolini Bussi (1996): mathematical discussion is a polyphony of articulated voices on a mathematical object, which is one of the motives of the teaching - learning activity. In this case the motive of the discussion is a specific theorem together with the idea of theorem itself (see below). Therefore the complex of conjecturing, arguing, proving and systematizing proofs related to a specific problem situation is taken into account by the teacher by means of mathematical discussion. The continuity between argumentation and proof is naturally emphasized by argumentative behaviours, but at the same time the distance between argumentation and proof (Balacheff, 1988; Duval, 1991; Moore, 1994) is taken into account by the teacher's careful management of discussion with the specific aim of the social construction of the sense and value of a theorem (Bartolini Bussi & Bergamini, 1996). Concerning this issue, we believe that two crucial points emerge from current literature: on the one hand, the problem of the motivation to proof; and on the other, the distinction between argumentation and mathematical proof. These two aspects are linked to each other in a complex way.

Motivation to proof can be expressed at different levels. At the first level the truth of the fact is central: Is a fact true? At the second level, truth may no longer be in question, but a foundation of truth is needed: Why is a fact true? Hence the sense and the need for this grounding process (validation) is detached from the truth of the fact. In the first question, the truth of the fact is uncertain whilst in the second the truth of the fact may be certain. In our opinion, the uncertainty status of the truth of a statement is crucial for the initial construction of the meaning of theorems and calls for the careful selection of problem-solving situations, where the production of a conjecture is required. A third level, which is not considered in the research studies reviewed in this paper, concerns the release of theorems from the issue of truth search. In other words, we do not deal with formal proofs and their release from semantics.

Within this general framework, we introduce two specific theoretical constructs, the 'cognitive unity' and the 'mathematical theorem', which may help the management of class work on geometry theorems, the functioning of problem-solving situations and the interpretations of student behaviour. These constructs may also represent instruments for analysing some difficulties students meet when following the traditional school approach to geometry theorems.

### 2.1. Cognitive Unity

Analysis of work done by past and present geometers highlights the continuity that exists between the process of statement production and the construction of its proof, as well as providing meaningful examples. This continuity is not evident at all in the theoretical systematization of ancient classical geometers such as Euclid and Apollonius, but is emphasised as from the 17th century, in documents that reveal the process by which a result has been obtained (Barbin, 1988). What is in play is the relationship between conjecturing and looking for a proof, in particular specifying the objects of the conjecture and determining stricter hypotheses or stating a new weaker conjecture (Alibert & Thomas, 1991; Lakatos, 1976; Thurston, 1994). More generally, the development of the relationship between conjecturing and proving witnesses the longstanding process of elaboration of the idea of rigour.

Does a cognitive counterpart of this analysis exist? A metaphorical definition may be useful in analysing student processes. The continuity between the processes of conjecture production and proof construction, recognizable in the close correspondence between the nature and the objects of the mental activities involved, expresses a cognitive phenomenon, which will henceforth be referred to as 'cognitive unity'. Some hints about 'cognitive unity' are given in Harel's investigation into student behaviour (Harel, 1996).

### 2. 2. Mathematical Theorem

However, in mathematicians' mathematics the aforementioned continuity between statement and proof is always considered in a theoretical context, even if the context can change over time; the existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense. Principles and deduction rules are intimately interrelated so that what characterises a mathematical theorem is the system of statement, proof and theory. Historical-epistemological analysis highlights important aspects of this complex link and shows how it has evolved over the centuries.

## 3. Towards Teaching Experiments

According to the theoretical framework presented in the previous section, two crucial elements characterize the approach to geometry theorems in our teaching experiments: the function of a particular field of experience, and the role of the teacher as a cultural and cognitive mediator.

Every field of experience has to be analysed in terms of limits and potentialities in fostering cognitive unity and a systemic approach to geometry theorems. Historical and epistemological analysis has allowed us to identify the following criteria which, in the presence of a culturally relevant piece of mathematical knowledge, make it possible to choose a field of experience and particular activities within it: the need for concrete and semantically pregnant referents that promote dynamic processes; and the availability of tasks, meaningful to the field of experience, that foster cognitive unity. Dynamic exploration of the problem situation can determine the production of conditional statements and the construction of proofs, with strong functional relationships between these processes (Boero et al., 1996; Garuti et al., 1996; Bartolini Bussi & Bergamini, 1996). The conditional form of statements, from Euclid to the present day, represents the functional connection between statement and proof: actually, a proof develops, in the form of a deductive chain, the link (which is implicit in the statement) between facts that are assumed as starting points in the frame theory and the 'thesis' of the theorem, under some conditions that are given as 'hypotheses'.

As far as the role of the teacher is concerned, we assume that the process of construction of the meaning of theorems, although rooted in the field of experience, requires cultural and cognitive mediation. Actually, the teacher is responsible for introducing pupils to a theoretical perspective, which, although not spontaneous, is needed for a systemic view of mathematical theorems. In our teaching experiments, the construction of a theory is pursued in the form of accepted principles: invariants in perspective representation; the evident properties of shadows produced by vertical nails; and the underlying logic of Cabri.

The research methodology is typical of long term teaching experiments: classrooms are observed for several months (or even years), by collecting individual texts and transcripts of collective discussions, together with teacher's reports. The length of the process determines evolution in the general assumptions, until specific hypotheses are reached. The specific aim of these studies is on the one hand to single out the conditions under which students can approach geometry theorems, and on the other to study the mental processes involved in such an approach. For these reasons, the direct and productive involvement of teachers in all the phases of research is called for in each of the three experiments. In spite of the common features they share, the studies deal with different didactic problems, and actually concern different school levels (5th, 8th and 10th grades). This requires completely different approaches to geometry theorems for two reasons: the different levels of cognitive development and geometrical knowledge that pupils have reached (geometry is taught in Italian schools from the 1st grade); and the general attitude towards mathematics and its methods derived from their past experiences. At the outset of work on geometry theorems, younger students do not yet have a sufficient grasp of geometry notions. For them, the approach to geometry theorems is a fundamental step in the process by which geometry gradually becomes a 'field of experience' (Boero et al., 1995) and a corpus of mathematical knowledge as well. At high-school level, where students have a grounding in geometry, the problem is how to manage the delicate relationship between their geometrical background and a new approach to this knowledge from a deductive point of view.

### 3.1. An Historical Digression: the Birth of a Theory

The history of geometry gives meaningful examples of the development of fully-fledged theories from a long standing tradition of spatial practices. In this section we shall explore a paradigmatic example: the birth of projective geometry from the long-standing process of assuming properties of space and vision as axioms and modelling definitions, and of proving practical rules of painting as theorems. Natural perspective was developed from the classical age (Euclid's Optics) onward with the aim of representing objects with illusionistic effects. Practical rules for painting were transmitted in artist's workshops and collected in treatises of practical geometry. In the 15th century natural perspective gradually gave way to artificial perspective. This was based first of all on the idea of the (central) vanishing point or point of flight: if we consider the picture plane as a vertical window the spectator stands in front of, the central vanishing point is the point of the picture plane where a line from the spectator's eye, ortogonal to the picture plane, cuts it. This definition is taken from a more recent treatise (by Brook Taylor, 1715) where the genesis from practice was already somewhat hidden. The genesis is more evident if we consider that in early treatises, which contain also a theory of vision in space (e.g. Piero della Francesca, 1464) the central vanishing point was named 'eye'. The history of the theoretical development of artificial perspective up to projective geometry is actually the history of its progressive independence from painting practices, from Desargues' first introduction of invariants (Field & Gray, 1987) to the 18th century treatises of linear perspectives of Brook Taylor and Lambert (Bessot & Le Goff, 1992): the incidence axioms listed by Brook Taylor gave birth to a projective approach to problems, and Lambert's use of perspective to prove properties of plane configurations stated definite autonomy from painting. Within the theory of projective geometry, based on incidence axioms, practical rules of painting assumed the status of theorems.

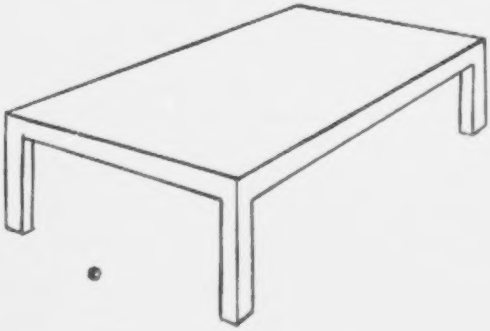
A similar analysis could be made for the genesis of other theories in the history of geometry (see the analysis of sunshadows in Serres, 1993 and the analysis of geometrical construction in Lebesgue, 1950). Actually, the above perspective has guided our experimental research studies into the school approach to geometry theorems.

## 4. A Survey of the Teaching Experiments

The teaching experiments that have been carried out by our research groups concern the approach to geometry theorems (production of statements and construction of proofs inside a frame theory) in three different fields of experience: the representation of visible space by means of geometrical perspective; sunshadows; and geometrical constructions in the Cabri - environment.

## 4.1. The Representation of Visible Space by Means of Geometrical Perspective

This experiment concerns the field of experience of the representation of the visible world by means of geometrical perspective. The early approach to theorems takes place in the educational context of 'mathematical discussion' as a fundamental tool for the social construction of knowledge. The most meaningful problem-solving situation, used in several 5th and 6th grade classrooms, concerns the drawing of a small ball in the centre of a table, drawn in perspective. In particular we shall refer to data from an experiment carried out with the same class over four years, from grade 2 to grade 5. Yet before explaining this problem, some information about previous school activities is needed (a more detailed account of which is contained in Bartolini Bussi, 1996). At the beginning of the whole experiment most of pupils (2nd graders) were able to draw simple three dimensional objects, such as a set of boxes, tables and chairs. The level of mastery was variable: a teaching experiment on the coordination of points of view (Bartolini Bussi,1996) had contributed to solve some of the early problems of real life drawing ( the drawing of hidden lines; the drawing of a base line for all the elements lying on the ground plane). Anyway no explicit reference to techniques of perspective drawing had previously been made in the classroom. This is the school context where the crucial problem situation was given. A table, drawn in perspective, was given together with the task: *'Draw a small ball in the centre of the table. You can use instruments. Explain your reasoning'.*



The strategies from the classroom can be roughly divided into three categories:

a) ROUGH ESTIMATE BY SIGHT: the ball is put directly on the table, without comment.

b) MEASURE-DEPENDENT: the point is chosen by means of measuring and tracing lines.

c) ALIGNMENT-DEPENDENT: the point is the intersection of diagonals.

Solution a) refers to everyday practice; solution b) is an attempt to rationalise by means of a scientific tool (measuring); while the correct approach, solution c), depends on an invariant of perspective representation, i.e. alignment of points. The solution based on rough estimation is predominant among young pupils such as 2nd graders, while older pupils prefer measure-based solutions as more precise (when compared to rough estimate) and therefore true. The alignment-dependent solution is the least popular even with older students up to university level.

In the 2nd grade class of the experiment, there was no discussion about this problem or the different solutions proposed by pupils. Later a tool of semiotic mediation was built to direct perspective drawing as well as to evaluate perspective representations (Ferri 1993). The following table shows the scheme built by the same class one year later (3rd grade): the geometrical properties of three-dimensional figures (still alluding to real objects because of the age of the pupils) and the corresponding properties of two-dimensional representations were compared.

| REALITY | REPRESENTATION |
|---|---|
| *several points of view* | *a fixed point of view* |
| *straight lines* | *straight lines* |
| *there is alignment* | *there is alignment* |
| *4 congruent angles* | *different angles* |
| *square* | *quadrilateral* |
| legs of the table: | legs of the table: |
| *4 parallel vertical lines* | *4 parallel vertical lines* |
| top of the table: | top of the table: |
| *2 by 2 parallel lines* | *2 horizontal parallel lines and 2 not* |

The scheme was built collectively under the teacher's guidance and reflects the random order of students' observations: each student recorded it in his/her copybook. The crucial row is the following: *straight lines* in the first column correspond to *straight lines* in the second one. This property was stated on the basis of empirical observation of both objects and images. Later it was used systematically and autonomously by pupils both to check drawing output and to direct the drawing process itself. Yet this early scheme of invariants (and of variants as well) functioned also as a germ of a whole theory of perspective representation. In the same classroom, the problem of the table and small ball was posed again at 5th grade level, three years after the first attempt: in the meantime different activities about perspective drawing had been carried out in the classroom (Bartolini Bussi 1996) but the 'table and ball problem' itself had not been discussed any more. The second formulation of the problem was a bit different. With the same drawing, the following task was assigned: *'Draw the small ball in the centre of the table. Prove the method you have used'.*

Actually the (correct) solution to the 'table and ball problem' happens to produce a statement and a proof framed by the germ - theory of perspective embodied in the table of invariants *).

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*)The complex of the statement (which expresses a construction process) and the proof of the theorem can be given in the following terms:
STATEMENT: If a quadrilateral is obtained as an image of a rectangle in perspective, the image of the centre of the rectangle is given by the intersection of the diagonals of the quadrilateral.
PROOF: In perspective, straight lines are represented by straight lines; hence the intersection of the diagonals of the rectangle is represented by the intersection of the diagonals of the quadrilateral.

In the classroom, all the pupils (19) gave the correct solution, but not everybody succeeded in proving the method. Of those who succeeded, we distinguish between those who gave statement and proof (6 pupils); and those who gave statement, proof and the genesis of the theorem (5 pupils). The latter term refers to the more or less detailed reconstruction of the meaning of the theorem within the activity on perspective drawing carried out over three years in the classroom. Some pupils proved to be aware that the problem had already been set three years before, even though it was not exactly the same problem: in this case historical reconstruction or metacognitive explanations of one's own long- term processes were offered. Of the other pupils (8), 5 'proved' the method by stating that the other methods were wrong, while the last 3 simply gave the correct solution. Of course, due to the age of the pupils involved, we accepted diversions from a rigorous formulation provided the intuition of generalisation, the intuition of abstractness and the deductive structure were conserved. As an example, we report excerpts from the work of two pupils:

Federico: *'The right method is to draw diagonals because in any figure, in perspective or in plan, the intersection of diagonals gives the centre, because straight lines retain their characteristic and remain straight'.*

Laura: *'Since: 1) the table is in perspective; 2) measures change; c) opposite angles (i.e. angles connected by a straight line) remain opposite, by connecting them with straight lines, the centre can be found'.*

To interpret the data correctly, it should be noted that the words 'proof' and 'proving' ('dimostrazione' and 'dimostrare' in Italian) had not yet any institutional meaning in the classroom: the words had been used freely by pupils in collective discussions to mean the proposal of very safe arguments. The above data show that most of the pupils related the statement to the shared theory in order to produce the safest deduction; i. e., they were confident that, as the theory was true, the logical consequences of the theory were also true.

The previous summary shows that the field of experience of visible space representation using geometrical perspective, through the approach of mathematical discussion, offers a suitable context for introducing to very young pupils an early idea of theorem, namely a statement with a justification that refers to an accepted theory. While measurement shifts reasoning to the empirical level, a geometry without measurement can open up different possibilities of exploration. Hence, perspective provides a field of experience in which measurement is no longer the empirical instrument for the validation of the results. Even if some of the students' products are expressed in a bare language that refers to logical rules and not to control in reality (see Laura above), their idea of theorem is undoubtedly that of a true statement whose justification lies in the theory: justification is thought to be needed mainly because it is an astonishing statement, if compared with the more readily acceptable measure - dependent solution.

## 4.2. Sunshadows

This experiment concerns the field of experience of sunshadows. The most meaningful problem solving situation, used with several 8th grade classes, concerns the study of the parallelism of shadows of two non-parallel sticks, leading to the proof that the conjectured condition for parallelism is sufficient and necessary. In two classes, activities were organised according to the following stages (for further details, see Boero et al., 1996; Garuti et al.,1996):

**a)** Setting the problem : *'In the past years we observed that the shadows of two vertical sticks on horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and an oblique stick? Can the shadows be parallel? Sometimes? When? Always? Never? Formulate your conjecture as a general statement.'*

We note that the task explicitely suggests a piece of knowledge to be used as a postulate (in the Euclidean sense, as a 'property of space').

**b)** Producing conjectures (individually, or in pairs).

**c)** Discussing conjectures: the conjectures were discussed with the help of the teacher until statements of correct conjectures were collectively obtained that reflected the students' different approaches to the problem.

**d)** Arranging statements: through different discussions orchestrated by the teacher, the following statements,'cleared' of metaphors and more linguistically precise than those produced by students at the beginning, were collectively attained: *'If the sun rays belong to the vertical plane of the oblique stick, the shadows are parallel. The shadows are parallel only if sun rays belong to the vertical plane of the oblique stick '; 'If the oblique stick is on a vertical plane containing sun rays, the shadows are parallel. The shadows are parallel only if the oblique stick is on a vertical plane containing sun rays'*

**e)** Preparing proof; the following activities were performed:

- individual search for analogies and differences between one's own initial conjecture and the 'cleared' statements;
- individual task: *'What do you think of the possibility of testing our conjectures by experiment?'*
- discussion concerning the students' answers to the preceding question. During the discussion orchestrated by the teacher, students gradually realize that experimental testing is *'very difficult': 'one should check what happens in all the infinite positions of the sun and in all the infinite positions of the sticks'*. This stage of activity (3 hours) was planned in order to enhance the students' critical detachment from statements, motivate them towards proving, and make it clear that henceforth class work would concer the validity of the statement 'in general'.

**f)** Proving that the condition is sufficient.

**g)** Proving that the condition is necessary.

**h)**Final discussion, followed by an individual report about the whole activity.

The most interesting results concern:

- Analysis of the sunshadows field of experience, as the basis of pieces of knowledge and empirical evidence allowing the production of general statements and their justification. The sunshadows field of experience is a context in which students can naturally explore problem situations in different dynamic ways. In order to study the relationships between sun, shadow and the object which produces the shadow, one can imagine (and, if necessary, perform a concrete simulation of) the movement of the sun, of the observer and of the objects which produce the shadows. The sunshadows field of experience also offers the possibility of producing, in open problem - solving situations, conjectures which are: meaningful from a space geometry point of view; not easy to prove; and lacking the possibility of substituting proof with the creation of drawings.

- Experimental evidence of the relevance of the dynamic exploration of the problem situation both in conjecture production and in the construction of proofs. The following hypotheses were confirmed by the teaching experiment.

As to conjecture production:

*A) the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, which will be subsequently detached from it and then 'crystallized' from a logic point of view* ('if... then..').

As to proof construction:

*B) for a statement expressing a sufficient condition, proof can be the product of the dynamic exploration of the particular situation identified by the hypothesis;*

*C) for a statement expressing a sufficient and necessary condition, proving that the condition is necessary can be achieved by resuming the dynamic exploration of the problem situation beyond the conditions fixed by the hypothesis.*

It seems to us that the students' collected texts clearly reveal that dynamic exploration of the situation singled out by the hypothesis fulfils an important function in promoting the logical connection between the property accepted as true (parallel sticks produce parallel shadows) and the property to be confirmed (shadows are parallel): movement of the stick keeps the direction of its shadow (since it happens in the vertical plane containing sun rays) and, therefore, opens up the possibility to reason in a transitive way (e.g.: the real vertical stick produces a shadow parallel to the one of the imaginary vertical stick; the oblique stick produces a shadow aligned with that of the imaginary, vertical stick; so the oblique stick produces a shadow parallel to that of the real vertical stick).

- Experimental evidence of cognitive unity between the phases of conjecture production and proof construction through the link revealed between the dynamic exploration in conjecture production and the construction of proofs. We detected a process with the following characteristics: *during production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices. During the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ('arguments') produced during the construction of the statement according to a logical chain.* Actually, as concerns the production of the statement, argumentative reasoning fulfilled a crucial function: it allowed students to consciously explore different alternatives, to progressively specify the statement and to justify the plausibility of the produced conjecture. On the other hand, students who produced wrong conjectures later showed the need to reconstruct the valid conjecture in order to produce the proof. The fact that poor argumentation during production of the statement always corresponded to lack of arguments during construction of the proof seems to confirm the close connection that exists between production of the conjecture and construction of the proof. Moreover, consistency between personal arguments provided during the production of statements and the ways of reasoning developed during the proof seems to be confirmed by two facts: I) the type of argumentative reasoning made during the production of the statement by one student was resumed by him/her (often with similar linguistic expressions) in the justification of the statement subject to proof; and II) the kind of dynamic process (movement of the sun or the stick) recorded at the conjecture stage was almost always the same as the one used at the proof stage. Yet, the dynamic exploration implemented during the construction of the proof, though remarkably similar to that implemented during the production of the conjecture in respect to the type of movement, differs deeply as to the function assumed in the thinking process: from a support to the selection and specification of the conjecture, to a support for the implementation of a logical connection between the property assumed as true (*'vertical sticks produce parallel shadows'* ) and the property to be validated.

### 4.3. Geometrical Construction in the Cabri Environment

This experiment concerns the field of experience of geometrical construction. The main point was to introduce pupils to the deductive approch to geometry, i. e. to the construction of a system of geometrical facts, coherently related according to the choice of primitives (axioms) and the method of deduction.

The history of the classic impossible problems (which puzzled the Greek geometers) tells us about the fundamental theoretical importance of the notion of construction (Henry, 1993). Despite the fact that there is a concrete counterpart to geometrical construction which can be accomplished on a sheet of paper, geometrical constructions have a theoretical meaning that overcomes the apparent practical objective. The tools and their rules of use correspond to axioms and theorems of a theoretical system; a construction given there is a theorem validating it, i.e. it states the relationships between the elements of the geometrical figure, which is represented by the drawing produced.

The complex relationship (Heath, 1956, pp. 124-31) between constructions and theorems is not immediate and is difficult for students to grasp. A drawing is a material product of concrete operations and its correctness is definitely controlled by empirical evaluation; theoretical control is not spontaneously achieved, but can result from the activities that pupils perform within the chosen field of experience (Mariotti, 1996). The main motive is the evolution of the idea of construction from the empirical to the theoretical level and the evolution of the justifying process from generical argumentaion to proving.

The main elements characterizing the project are: the semiotic mediation (Vygotskij, 1978) offered by the Cabri environment (the primitive commands and macros force students to make explicit geometrical properties hidden in free-hand drawing); the dragging function which starts as a control tool to check the correctness of the construction, then becomes the external sign of the theoretical control; mathematical discussion as a basic element in the social construction of the meaning of theorem.

### 4.3.1. The Cabri Enviroment

In the Cabri environment, the construction activity, i. e. drawing figures through the available commands on the menu, is integrated with the dragging function: in other words, the construction of a figure can be associated with a control by dragging. Thus a construction task is solved if the figure on the screen passes the dragging test. In this case, the necessity of a justification for the solution comes from the need to explain why a certain construction works (i. e., it passes the dragging test); thus, a justification comes from the need to validate one's own construction, in order to explain why it works and/or foresee that it will do so.The key point is that what must be validated is the correctness of the construction; i. e., it is not the product of a procedure that must be validated, but the procedure itself; the necessity of this validation is mediated by the necessity of explaining why the Cabri-figure will not be messed up. When dragging is used, why do some constructions work and not others? The dragging function is accepted as a validating tool, but the problem must be shifted from validating by dragging to explaining the 'proof by dragging' itself. According to the theory of figural concepts (Fischbein, 1993; Mariotti, 1992, 1995a, 1995b), pupils must achieve the conceptual control over what they see on the screen. It is this very control that we want to promote, together with the idea of a coherent system. The richness of the environment may emphasize the ambiguity about intuitive facts and theorems and may constitute an obstacle to the choice of correct hypotheses. Thus, the basic idea of working inside a microworld was adapted to our objective. At the beginning, an empty menu is presented and the choice of commands discussed, according to specific statements selected as axioms, then the other elements of the microworld are added, according to new constructions and in parallel with corresponding theorems. In this way the system is slowly built up; step by step the complexity increases. The analysis of pupils' texts shows the slow evolution in the meaning of construction. At first, a construction is conceived as a concrete process towards production of a drawing, which has its justification in the acceptability of the product; then, a construction is conceived as a theoretical procedure which has its justification in a theorem. On the one hand, the descriptions of the procedure change, improving clarity through increasing mastery of correct terms; on the other hand, the argumentations approach the status of theorems, i. e. the justifications provided by the pupils assume the form of a statement and a proof.

### 4.3.2. Mathematical Discussion

According to the general hypothesis about social construction of knowledge, mathematical discussion (see Section 2) plays an essential part in classroom activities. In this case, the motive concerns the evolution of the personal senses of justification, related to the problem of construction. The cognitive dialectics between personal senses and the general meaning (which is constructed and promoted by the teacher) concerns the senses of justification and the general meaning of mathematical proof. Different senses of justification correspond to different goals of discussion, so that there is an evolution from a first goal:

to determine the epistemic value (Duval, 1991) of a certain fact or statement (Is the figure a square? Is the figure always a square? Why?), to a second goal:

to determine the correctness of an argumentation according to the stated criteria (What is the underlying theorem?).

The second goal is related to a general motive of activity, i.e. awareness of a new theoretical status of certain statements and of their relationships within a theory. The development from the first to the second goal can be described as follows:

| | | | | |
|---|---|---|---|---|
| Is the drawing correct? | | Is the procedure correct? | | Is the theorem valid? |
| | ---------> | | ---------> | |
| Empirical observation | | Argumentation | | Proof |

At the beginning, the focus of the discussion is on the comparison of the different drawings produced; it then moves on to the comparison of different procedures. This is the first crucial change required to finally approach the heart of the problem: validation in terms of a system of definite rules. In each phase the role of the teacher is fundamental in order to move the goal of the discussion and to guide the evolution of the personal senses towards the geometrical meaning of a construction problem. Moreover, in each phase the reference to Cabri is fundamental; passing from empirical control of the drawing to validation of the procedure, reference to the Cabri-figure and the dragging function is vital. The specificity of the Cabri environment clarifies the question about the procedure, rather than about the product. Without the mediation of the Cabri environment this sense may be unintelligible. When the drawing is done on a sheet of paper, it is very difficult to avoid validation of the construction focusing on the drawing itself and on direct control by observation. Actually, the question is about the drawing, but its sense concerns the geometrical figure that it represents, thus the ambiguity between drawings and figures constitutes an obstacle. On the contrary, in order to grasp the theoretical status of the question, a crucial factor is the reference to a 'microworld' which embodies a theory, that has its own independence (the machine has its own logic); and which is external to the subject (but is accessible to the pupil via the dragging function). In addition, the microworld is also independent from the teacher's authority and promotes a personal relation of the learning subject with geometrical activity (changing the didactic contract, Brousseau, 1986). Looking for a justification within the system of rules of the machine (in this case, the Cabri environment) introduces pupils to a theoretical status of justification: thus it is not the discussion in itself, but the discussion guided by the teacher according a specific goal which determines the meaning of the construction problem to evolve from the empirical to the theoretical level .

## 5. Conclusions

First of all, we consider in what sense during our experiments students have performed a mathematical activity concerning theorems. This problem is particularly relevant for the first and the second experiment. The objects of these experiments were hypotheses concerning the rationalisation of drawing practice and the physical phenomenon of sunshadows; they had as geometric counterparts, at the model level, a statement of central projection and a statement of parallel projection geometry. In the first experiment, pupils produced conjectures about drawing, collected them in the table of invariants and proved to be able to use those conjectures (assumed as postulates) to justify a process with solid arguments. In the second experiment, students produced their conjecture as a hypothesis concerning the phenomenon of sunshadows; when they verified their conjecture most of them seemed to be aware of the fact that they had to get to the truth of the statement by reasoning, starting from true facts. Most of them produced a validation realized through a deductive reasoning. Actually their reasoning started from properties considered as true, (such as 'two vertical sticks produce parallel shadows') and got the truth of the statement in the 'scenario' determined by the hypothesis. In this way, students produced neither statement of geometry in the strict sense, nor a formal proof: objects were not yet geometric entities, deduction was not yet formal derivation. But their deductive reasoning shared some crucial aspects with the construction of a mathematical proof. In the third experiment, in spite of the presence of concrete referents (drawings in the Cabri environment), axioms and modes of deduction were in the foreground.

Moreover, the whole activity performed by students in all the experiments shared many aspects with mathematicians' work when they produce conjectures and proofs in some mathematics fields (e.g. differential geometry): mental images of concrete models are frequently used during those activities. As to proof, mathematicians frequently come close to realising the ideal of the formal proof only during the final stage of proof writing. During the stage of proof construction, the search for 'arguments' to be 'set in chain' in a deductive way is frequently performed through heuristics, reference to analogical models and taking into account the semantics of considered propositions (cf Alibert & Thomas, 1991; Thurston, 1994). For these reasons we think that the activities performed during our teaching experiments may represent an approach to mathematics theorems which is correct and meaningful from the cultural point of view. In this way, our experiments also show that the contradiction highlighted by Duval between everyday argumentation and deductive reasoning, between empirical and geometrical knowledge, can be managed in dialectic terms within the evolution of classroom culture. This can be achieved through the distinction between the nature of the productive process and the formal features of the final product, on the one hand and between the 'concreteness' of the referents in the productive process and the conventionality and abstraction of geometrical objects on the other.

In all three experiments classroom culture is strongly determined by recourse to mathematical discussion orchestrated by the teacher to change the spontaneous attitude of students towards theoretical validation. In our research studies, this general conclusion is related to the study of the conditions and processes that allow students' constructive approach to geometry theorems.

Features of the contexts were analysed as sources of 'concrete' referents and dynamic processes for the production of statements and the construction of proofs. Each context allowed and encouraged students to develop specific processes of dynamic exploration and promoted work with spatial metaphors, which are essential in the production of geometry theorems.

Features of dynamic exploration were analysed to detect different functions. In all three fields of experience, the initial outcome is the generation of a space of possible configurations to be explored with different goals. In the field of experience of representation of visible space, dynamic exploration contributes to the collective construction of the theory (table of invariants) and to solution of the problem. In the sunshadows field of experience, dynamic exploration allows production of the conjecture and construction of the proof in the subspace defined by the hypotheses. In the field of experience of geometrical constructions in the Cabri environment, the need to produce figures that resist dynamic exploration (dragging) is the driving force behind the production of acceptable constructions and proof of their validity.

The hypothesis of 'cognitive unity' on which we worked seems to have important didactic implications, since it calls into question the traditional school approach to theorems. In fact, teachers in Italy, as well as in other countries, usually ask students to understand and repeat proofs of statements that they supply; this appears one of the most difficult and selective tasks for grade IX-X students. Only as a possible final stage (often reserved to top-level students or students choosing an advanced mathematical course) does the teacher ask the students to prove statements, generally not produced by students but suggested by the teacher. Even more seldom are students involved in the process of producing conjectures. If our hypothesis is valid, during this traditional path student difficulties may at least partly depend on the fact that they have to reconstruct the cognitive complexity of a process in which mental acts of different nature functionally intermingle, beginning with tasks that by their nature lead towards partial activities that are difficult to reassemble in a single whole.

## 6. Some Emerging Problems

In this paper we discuss the problem of approaching geometry theorems at different age levels, within the same theoretical framework. Our findings confirm the possibility of early introduction to theorems in suitable fields of experience, provided a suitable epistemological analysis is performed.

The problem of transferring the capabilities developed in our fields of experience to other "purely mathematical" contexts remains to be tackled. With regard to this problem, later observations made while the two classes involved in the sunshadows experiment (Section 4.2.) worked on traditional geometry theorems produced some evidence of transfert. In particular, many students (of both high and average level) could imagine the dynamic exploration of the geometric figures proposed for the formulation of conjectures and proofs.

Another research problem concerns the epistemological analysis of statements from the point of view of their conditional form, and its relation to the production process. As conditional form is crucial in the production of theorems, for a given statement we need to look for contexts and tasks (within contexts) that can induce this form. We are well aware that dynamic exploration is not the only possible source of this form. For an example, see Boero & Garuti (1994): during a teaching experiment concerning production of statements, the following type of reasoning was identified in 3 students out of 34: *"The length of the shadows is proportional to the height of the sticks due to the parallelism of the sun shadows .... If the lines are parallel, the lengths of the segments cut on another two lines will be proportional"*. In this case the student passes from a recognition of causal dependency between parallelism and proportionality in the physical phenomenon, to the conditional statement that takes into account the possibility that lines cannot be parallel.

Different fields of experience are related in the literature to the issue of geometry proofs: sunshadows (4.2.); geometric construction in Cabri (4.3.) or other software (e.g. Geometric Supposer or Geometer's Sketchpad in the perspective of the 'dynamic geometry' indicated by Goldenberg & Cuoco, 1995); the 'mathematical machines' (Bartolini Bussi, 1993), gears and mechanisms (Bartolini Bussi et al., submitted) and the 'representation of the visible space' (4.1). Further analysis and specific comparison needs to be done concerning the potentialities and limits of those contexts in the genesis of conditionality and the transition from justification to proof.

## References

Alibert, D. & Thomas, M.: 1991, 'Research on mathematical proof'. In D. Tall (Ed.), *Advanced Mathematical Thinking*, Kluwer Ac. Pub.

Balacheff, N.: 1988, *Un Etude des processus de preuve en mathématique chez des élèves de College*. Thèse d'Etat, Université Joseph Fourier, Grenoble

Balacheff, N.: 1991, The Benefits and Limits of Social Interaction: the Case of Mathematical Proof, in *Mathematics Knowledge: Its Growth through Teaching*, Kluwer Acad. Pub.

Barbin E., 1988: 'La démonstration mathématique: ....', *Bulletin APMEP*

Bartolini Bussi M. : 1993, 'Geometrical Proof and Mathematical Machines: An Exploratory study', *Proc. of PME-XVII, Tsukuba*

Bartolini Bussi M. & Bergamini B.:1996, *The Theorems of Sun*, paper presented at V Encontro de Investigacao em Educacao Matematica, Troia (Portugal);

Bartolini Bussi M.:1996, 'Mathematical Discussion and Perspective Drawing in Primary School', *Ed. Studies in Math.*

Bartolini Bussi & al. (submitted), *'Gears as a Field of Experience between Real Life, Technology and Geometry'*

Bessot D. & Le Goff J. P. : 1992, 'Mais où est donc passée la troisième dimension?', in *Histoires de Problemes, Historie des Mathématiques*, Lyon: Commission Inter - Irem

Boero P. and Garuti R.:1994, 'Approaching Rational Geometry ...',*Proc of PME XVIII*, Lisbon

Boero, P.; Garuti, R.; Mariotti, M.A.: 1996, 'Some dynamic mental processes underlying producing and proving conjectures', *Proc. of PME-XX*, Valencia

Boero, Paolo et al.: 1995, 'Aspects of the Mathematics-Culture Relationship in Mathematics Teaching-Learning in Compulsory School', *Proc. of PME-XIX*, Recife

Brousseau, G.: 1986, *La théorisation des phénomènes d'enseignement des mathématiques*, These, Université de Bordeaux-I

Costa C., Ferri F. & Garuti R.: 1994, 'Perspective drawing as a semiotic tool towards the statements of geometry for young pupils (grades 3-6)', *Proc. of CIEAEM-46*, Toulouse

De Villiers, N.: 1991, 'Pupils' neeeds for conviction and explanation...', *Proc. of PME-XV*, Assisi

Dreyfus T.: 1991, 'On the Status of Visual Reasoning in Mathematics...',*Proc. of PME-XV*, Assisi

Duval, R.: 1991, 'Structure du raisonnement deductif ...', *Ed.St. in Math.*

Ferri F.:1993, 'Rappresentazione del mondo visibile', *Scuolaviva*

Field J. V. & Gray J. J.:1987, *The Geometrical Work of Girard Desargues*, Springer - Verlag

Fischbein E.: 1993, The theory of figural concepts, *Ed. Stud. in Math.*

Garuti, R.; Boero, P.; Lemut, E.& Mariotti, M.A.:1996,'Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems', *Proc. of PME-XX*, Valencia

Goldenberg, E.P. & Cuoco A.: 1995, *What is Dynamic Geometry?* , paper presented at the Int Conference 'Justifying and Proving in School Mathematics', Inst. of Education, London

Hanna, G. & Jahnke,H. N.: 1993, 'Proof and application', *Ed. Studies in Math.*, 421-438

Hanna, G.: 1996, 'The ongoing value of proof', *Proceedings of PME-XX*, Valencia

Harel, G.: 1996, 'Classifying processes of proving', *Proceedings of PME-XX*, Valencia

Heath, T.:1956, *The thirtheen Books of Euclid's Elements*, Dover, vol. 1.

Henry, P.:1993, 'Mathematical machines', in *The machine as metaphor and tool*, Springer- Verlag

Laborde, C.: 1992, 'Solving problems in computer based geometry environment: the influence of the feature of the software', *Zentr. fur Did. der Math.*

Laborde, C.: 1993, 'The computer as part of the learning environment: the case of geometry', in *Learning from computers: mathematics education and technology*, Springer Verlag

Lakatos, I.: 1976, *Proofs and refutations*, Cambridge University Press

Lebesgue H.: 1950, *Leçons sur les Constructions Géométriques*, Gauthier-Villars

Maher, C.A.: 1995, *Children's Explorations Leading to Proof* , paper presented at the Int Conference 'Justifying and Proving in School Mathematics', Inst. of Education, London

Mariotti, M.A.:1992, 'Geometrical reasoning as a dialectic between the figural and the conceptual aspect', *Topologie structurale / Structural topology*

Mariotti, M.A.:1995a, 'Images and concepts in geometrical reasoning', in *Proc. of the Nato A.R.W. 'Exploiting Mental Imagery with Computers in Mathematics Education'*, Springer-Verlag

Mariotti M.A.:1995b, *'Introducing Pupils To Proof: A Teaching Experiment'*, paper presented at the Int. Conference 'Justifying and Proving in SchoolMathematics', Inst. of Education, London

Mariotti, M.A.: 1996, 'Costruzioni in geometria', *L'insegnamento della Matematica e delle Scienze Integrate*

Moore, R. C.: 1994, Making the Transition to Formal Proof, *Ed. Studies in Math.*

Serres M: 1993, *Les Origines de la Géométrie*, Flammarion

Sfard A.:1991, 'On the dual nature of mathematical conceptions: reflections on processes and objects as differents sides of the same coin', *Ed. Studies in Math.*

Simon, M.:1996, 'Beyond inductive and deductive reasoning', *Ed. Studies in Math.*

Thurston, W. P.:1994, 'On proof and progress in mathematics', *Bulletin of the A.M.S.*

Vygotskij L. S.:1978, *Mind in Society. The Development of Higher Psychological Processes*, Harvard University Press

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# APPROACHING GEOMETRY THEOREMS IN CONTEXTS: FROM HISTORY AND EPISTEMOLOGY TO COGNITION

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The authors have attempted an ambitious project in designing three teaching experiments to investigate different contexts for geometry theory development; the role of the teacher in classroom interaction and the evolution of theorems in terms of their initial conjecture, formulation and eventual proof (explanation). Although they have clearly widely consulted the available literature in the area, the theoretical framework unfortunately appears a little unfocussed and haphazard at times.

The authors correctly identify students' motivation to proof as a major problem and distinguish two important functions of proof, namely, *verification* (dealing with the truth of a statement) and *explanation* (dealing with understanding why it is true), but neglect to distinguish the *discovery* function (compare De Villiers, 1995). They appear to distinguish two hierarchical levels with verification at the first level and explanation at the second level; in other words, understanding the verification function appears to be a prerequisite for understanding the explanation function. For example, they claim that "*the uncertainty status of the truth of a statement is crucial for the initial construction of the meaning of theorems*". This is problematic as I believe that these functions may vary dramatically from context to context, and according to students' general cognitive development and experience. For example, letting students construct a kite using reflection on Sketchpad or Cabri in order to investigate its properties may lead more naturally to their *explanation* via symmetry, than to their actual *verification*, as students tend to have little doubt having dynamically investigated them. Of course, it is extremely important to develop this verification meaning of proof amongst students and that one identifies appropriate contexts to do so, but it is potentially restrictive to regard it as a necessary prerequisite for an initial introduction to proof.

For example, consider the excerpts from the work of the two students in relation to the "*table-and-ball*" problem. In my opinion, these excerpts can just as easily be interpreted as their *explanations* for *why* their solutions are correct. It is certainly not conclusive that the students themselves interpreted their arguments purely as *justifications*, which is what the researchers seem to want to attribute to them. In order, to try and assess the meaning students give to their own arguments, the researchers should have attempted to formulate the question differently (eg. to ask students to "*Prove the method ...*" already assigns an *a* *priori* meaning of *justification*, since it means "*the proposal of very safe arguments*"). Alternatively, they could have probed deeper by asking further questions, for example: "*Are you convinced that your solution is correct? Why?*"and "*Can you explain why your solution is correct?*" It is possible to conjecture that students would interpret their own arguments as both justifying and explaining at the same time.

It is also not clear whether students had any prior experience in solving similar real world problems; eg positioning objects on an actual table or a small model of it (not just a representation). Either way, it would be important to try and determine what the effect of the presence, or absence, of such prior experience might be on the outcome. For example, one may find that in the real world situation some students may instead use the lines of symmetry (through the midpoints of the opposite sides) of the rectangular shaped table to locate the centre. There is furthermore not enough detailed information to assess the transition from reality to the presentation of reality by perspective drawing, and the precise role of the teacher and the involvement of the students in this process.

In the Sunshadows paragraph on the 9th page, the researchers refer to "*proving that a condition is necessary*". Logically, this is a little confusing, as proving strictly deals with establishing the *sufficiency* of certain conditions. For example, in proving a theorem $p \Rightarrow q$ true, we establish that the condition $p$ is sufficient for the conclusion $q$. We do not really "prove" conditions necessary in mathematics (unless the conditions are necessary **and** sufficient, in which case we have an equivalence and have to prove the foward and backward implications, ie. conditions are really minimal (necessary) by leaving out some of them. For example, for a quad to be a rhombus it is clearly a sufficient condition to have diagonals perpendicularly bisecting each other. To explore whether this is also a necessary condition, one could try leaving out some of the conditions; eg check whether perpendicular diagonals or bisecting diagonals are sufficient on their own. It is this kind of exploration (rather than formal proving) that the researchers appear to be referring to on the 10th page under C).

A significant finding in the Sunshadows paragragh was the close connection between the production of the conjecture and the eventual construction of a proof. However, this finding may be context specific and one must be careful not to overgeneralize.

According to the researchers, the main purpose of investigating geometric constructions within a Cabri environment was to introduce students to an axiomatic-deductive approach to geometry. By selecting certain statements as axioms, the system is slowly built up by adding new constructions. This kind of axiomatization can be called *constructive* axiomatization where one starts out from a small number of axioms and logically deduce the one theorem after the other (compare Krykowska, 1971:129). However, from a historical perspective, Euclidean geometry did not initially develop in this way, but was only **re-organized** in this way by Euclid. The latter kind of axiomatization can be labelled as *descriptive* axiomatization (compare Krykowska, 1971:129-130) which means that **after** a certain set of statements have been already been discovered, known and used for a while, one starts analysing the logical relationships between them, first locally and then more generally. Finally, a subset of these statements are selected as axioms, and the remaining statements are re-structured into a deductive frame.

Following a descriptive axiomatization approach, one could therefore easily let students use angle bisector, perpendicular bisector and other tools within Cabri or Sketchpad **without** necessarily first posing them with the problem of designing the tool themselves, explaining why they work or proving that they work. In this sense, constructions could simply be viewed as tools to explore and investigate interesting geometric relationships such as the concurrency of the angle bisectors of a triangle, etc. At a later stage, one could then come back to the logical systematization of the underlying theory on which the construction tools are based. From an epistemological view, it is therefore not essential ("correct" as the researchers seem to claim) that one has to develop and introduce constructions in a constructive axiomatic way. More generally, one could develop a substantial body of geometric knowledge before axiomatizing locally, and then more generally (compare Freudenthal, 1973:451).

For example, one could first let students discover that the midpoints of the sides of a quad determine a parallelogram, and then use as explanation the (unproved) statement that the line segment connecting the midpoints of two sides of a triangle is parallel (and equal to half) the third side. Next one can again ask how we may prove this midpoint theorem in turn, and continuing to reason backwards in this way, one can arrive at appropriate axioms. Epistemologically, the researchers' analysis and implementation of the *different* ways in which mathematics develops and is systematized, therefore appears to be limited.

**References**

De Villiers, M.D. (1995). An alternative approach to proof in dynamic geometry. **MicroMath**, Spring, 11(1), 14-19.

Freudenthal, H. (1973). **Mathematics as an Educational Task**. Dordrect: D. Reidel.

Krygowska, A.Z. (1971). Treatment of the Axiomatic Method in Class. In Servais, W. & Varga, T. **Teaching School Mathematics.** Penguin-Unesco, London, 124-150.

# WORKING GROUPS

# UNDERSTANDING OF MULTIPLICATIVE CONCEPTS

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Mathematical concepts tied to "multiplicative conceptual fields" include multiplication and division, ratio, rate, fraction and rational number, linear functions, and linear mappings (Vergnaud, 1988). These concepts play a central role in school mathematics and their developments are closely connected (Harel & Confrey, 1994). This working group continues the discussion group of previous years and brings together researchers who have studied children's understandings of these concepts to share their findings and inform each other's future investigations. It is hoped that discussion will lead to a coordinated framework to approach future research.

This year, the intention is for the working group to focus on three aspects of this research. The first is theoretical perspectives which connect and provide a common or comprehensive framework for the various multiplicative concepts (e.g., extending Greer, 1992). It is hoped that some of these frameworks will relate to to learning and to metaschemes such as abstract schema (Ohlsson, 1993). The second is studies of children's understanding of multiplicativity in a variety of contexts (e.g., area of rectangles). It is hoped that some of these will relate to topics which do not appear directly related to multiplicativity (e.g., the numeration system). The third is studies related to instructional programs that offer potential for effective learning in the conceptual field.

The working group will offer participants the opportunity to present the findings of their research, to discuss the findings of other participants and to form groups for collaborative research. It is hoped to do this in conjunction with the website set up for this working group (www.fed.qut.edu.au/projects/mult_pme/). It is hoped that papers can be placed on the site before the conference to be read by participants, that the site will provide a forum for participants' research after the conference, and that papers will be collected into a publication.

## References

Greer, B. (1992). Multiplication and division as models of situations. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.276-295). New York: Macmillan.

Harel, G. & Confrey (Eds.) (1994). The development of multiplicative reasoning in the learning of mathematics. Albany, NY: SUNY Press.

Ohlsson, S. (1993). Abstract schemas. *Educational Psychologist, 28*(1), 51-66.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: National Council of Teachers of Mathematics.

## Geometry Working Group

The last two meeting of the Geometry Working Group (Recife, 1995, Valencia, 1996) were devoted to theme:

"Different external representations in the geometrical field: their dialectic relationship with geometrical knowledge."

It seems almost impossible to conceive geometry without 'figures', but the ambiguity of the term *figure* was often pointed out and focuses the deep link between the two aspects, the mathematical object and to its 'concrete' representation. The complex relationship between images and geometrical ideas constitutes both a power and a weakness for Geometry. Historic analysis shows the basic contribution to geometrical theorisation given by experiences and theories about graphical representations (Consider the case of Projective Geometry).
Recent research dealt in several different ways with the relationship between technical drawing and geometrical education. New software are now available, which provide screen images and open new perspectives on geometrical education.
Certainly, the presence of new technologies and the availability of purposeful software raise the problem of images anew; but, using new technologies in geometrical education does not exhaust the complexity of the problem. At the same time, although their presence adds new elements to the analysis, focusing to computers risks to hide the rich contribute coming from other sources. A common point of view from which all these different aspects can be approached is the analysis of the interaction between external representations and geometrical knowledge.

The activity of the Group will continue with the aim of collecting and discussing contributions on this theme.

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# A NEW RESEARCH PARADIGM FOR SOCIAL ASPECTS OF MATHEMATICS EDUCATION : THE EVOLUTIONARY PROCESS IN GENDER BASED RESEARCH.

The evolution of a new research paradigm for gender based research in a small group investigative problem solving situation with adult undergraduate students' provides the focus for this development in research design. The research methodology is student centred in approach in that the respondents themselves set the focus for the ethnographic interviews and the quantitative data collection whilst participant observers in the mathematical problem solving groups. From initial pilot unstructured interviews the participants determined what the relevant gender variables for the research should be. These gender variables then became the focus for the semi-structured ethnographic interviews. The responses from these interviews were then used inductively to make direct observations of the problem solving activities and for the collection of survey data by means of a questionnaire to provide quantitative data of a more generalizable nature. In addition life history accounts permitted an analysis of the persistence of gender variables over time and also facilitated the making inferences from the ethnographic data. This methodology thus permitted the in depth insightful nature of ethnographic research methods to be harnessed with more generalizable quantitative data. The dynamics of the evolution of such a research design provides a new paradigm for research into social aspects of mathematics education and the relative gain for validity of research findings are critically analyzed. One consequence of the new research paradigm has been the development of the use of contingency tables and appropriate statistical user of Chi-square tests to analyse the statistical data with a few small cell entries in a consistent manner that otherwise had not been possible.

Further details of the new research paradigm will be presented at the twenty-first international conference for the Psychology of Mathematics Education together with details and analysis of the research findings from the qualitative and quantitative data developed in the new research paradigm.

The use of inductively imposed multi-facetted research methods that react to the needs of the research project thus provides a new structure for appropriate research techniques that will tap the relative strengths of different research strategies and provide reliable and valid research findings, and facilitate the development of new research paradigms in the future for mathematics education.

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## Working Group: Advanced Mathematical Thinking

The AMT working group is concerned with all kinds of mathematical thinking, developing and extending theories of the Psychology of mathematics Education to cover the full range of ages. This interest includes gathering information on current research, discussions of both the mathematical and psychological aspects of advanced mathematical thinking, and research into thinking in specific subject areas within mathematics. This will be the twelfth meeting of the working group.

Our first session will begin with short reports of recent research of AMT members. There will also be a short summary of on-line discussions of interest to AMT members. After discussion of the reports, the group will split into two subgroups.

One subgroup will continue planning of the book project proposed in Recife and begun in Valencia. This book is intended for mathematicians teaching at the post-secondary level, and will include material on teaching practice, students' beliefs, concerns and conceptions about mathematics, and the character of successful and unsuccessful students at the post-secondary level. This book is intended to compliment the group's last publication (Advanced Mathematical Thinking, D. O. Tall, Ed.) by making current research accessible to the general population of teachers of post-secondary mathematics, in a form which can be easily applied to their practice.

The second subgroup will discuss and select topics for future work of the working group, so that work other than the book project can continue in parallel. Defining the nature of future meetings of AMT will be an objective of this group.

The second session will be a continuation of the work begun by the two subgroups in the first session.

On the third day we will reconvene as the full group to hear reports from each subgroup, as well as from the stochastics working group, whose interests overlap those of AMT. The session will close with planning of work for the coming year and at PME-22.

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# Working Group

## Algebraic Processes and Structure

Organizers:

Teresa Rojano, Matemática Educativa, CINVESTAV, México.
Luis Radford, Université Laurentienne, Canada.

Over the past years, this Working Group has been studying algebraic thinking from different perspectives.

In our last meeting in Valencia, we worked together on the potentiality of problems in the teaching of algebra and concentrated particularly on the role played by "real" and "non-real" problems in the development of students' algebraic methods and ideas.

One of the most salient points raised by some participants was that in order to fruitfully identify the role of "real" and "non-real" problems in the teaching and learning of algebra, it is necessary to relate the practice of both kinds of problems to concomitant social and cognitive variables –e.g. curricular particularities, students' affectivity, classroom communicative practices, discursive processes and cognitive factors.

This year, we want to further pursue our work in the previous direction and focus on "real" and "non-real" problems and their link to:

a) curricular themes
b) affectivity
c) engagement
d) cognitive complexity
e) realistic/non-realistic characteristics.

In order to do so, some participants will present a short talk that will be followed by a discussion. A booklet containing the short talks will be made available to the participants.

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## Research on the Psychology of Mathematics Teacher Development

The Working Group on *Research on the Psychology of Mathematics Teacher Development* was first convened at PME 10 in 1990 and has continued to meet in this format at the annual PME conferences since then. A strong feature of the Working Group has been its cohesiveness and its wide representation across many countries. At PME 21 we hope to build on the foundation of shared understandings that have developed over the past years

### Aims of the Working Group

The Working Group aims to

- develop, communicate and examine paradigms and frameworks for research in the psychology of mathematics teacher development;
- collect, develop, discuss and critique tools and methodologies for conducting research concerning the development of mathematics teachers' knowledge, beliefs, actions and reflections in order to better understand the process of teacher change;
- implement collaborative research projects;
- foster and develop communication between participants.

### Plans for Working Group Activities at PME in 1997

This year we hope to present the Working Group monograph *Research on Mathematics Teacher Development: An International Perspective* edited by the former convenor Nerida F. Ellerton who retired in 1996. This book focusses on the collection, development, discussion and critiquing of paradigms and frameworks for research in the psychology of mathematics teacher development.

The first Working Group session in 1997 will begin with an invited presentation by Konrad Krainer (Austria) on current international issues and research findings concerning the psychology of mathematics teacher development. His talk will be followed by a discussion of the Working Group's focus and aims for this years meetings. The second and third meetings will allow time for the realisation of these goals and the formation of sub-groups sharing a particular goal or concern. The third session will conclude with the planning of Working Group activities for the coming year and at PME 22.

Andrea Peter
University of Munster, Germany

Vania Santos
Federal University of Rio de Janeiro, Brazil

# WORKING GROUP ON THE TEACHING AND LEARNING OF STOCHASTICS

Carmen Batanero, University of Granada
John Truran, University of Adelaide
Kathleen Truran, University of South Australia

Stochastics teaching and learning is a growing field of research with important implications for classroom practice. This growth is likely to continue. During 1996/97 three major chapters or books related to this work are being published—on assessment, data analysis and probability learning. In recent years, several major bibliographies have been prepared, of which some have been made public and some have only private circulation.

It is the rapidly expanding literature in the field which is not being codified and so provides a formidable hurdle for new researchers to overcome before they undertake their own work. Furthermore, the work of some researchers who do not write in English is not reaching many workers.

This working group plans to develop and publish a short annotated bibliography of about 200 critical papers which will be able to serve as a starting point for entering the literature, and as a companion to the three major chapters.

Members of the working group, including those who are not attending the conference, will be asked to prepare a short list of what they see as the 20 most critical papers in their field and the Working Group sessions will be devoted to moulding this material into a unified whole.

# PME XXI WORKING GROUP

# CLASSROOM RESEARCH

Organizers: Draga Vidakovic, draga@unity.ncsu.edu
Lyn D. English, L.english@qut.edu.au

The purpose of this group is to examine issues and techniques related to research involving the learner in the classroom.

The topic of discussion will focus on the current issues and difficulties in gathering valid data in classroom research studies. Issues addressed will include different approaches to gathering classroom data, and the ways in which a segment of a classroom situation could be analyzed using different research frameworks and techniques.

Short presentations by few speakers will address the current issues and share their experiences in classroom research. Participants are invited to share their own experiences with the process of understanding and articulating the difficulties the researcher may face in conducting the classroom research. Also, the participants will be invited to work in small groups on analyzing a segment of a classroom situation and share their analysis with the whole group. Prior to the conference, participants are encouraged to submit to the organizers abstracts or drafts of papers that deal with their classroom research. The organizers hope to work towards publishing these papers in book form.

# CULTURAL ASPECTS IN THE LEARNING OF MATHEMATICS

Coordinators:
Norma Presmeg, Marta Civil, Phil Clarkson, and Judit Moschkovich

At the final meeting of this Working Group in Valencia, Spain, last year, it was decided that the group should continue, and that future meetings should have a dual focus, taking into account both the needs of newcomers to the group, and those of previous participants. The new Organizing Committee express thanks to Bernadette Denys and Paul Laridon for their able leadership.

The intent of the meetings in Lahti is to provide a relaxed forum (in a circle if possible) for rich discussions relating to issues surrounding culture in the learning of mathematics at all levels. Definitions will inevitably be revisited, and themes such as enculturation, acculturation, equity, and the role of language in learning mathematics will be addressed. Participants will be given the opportunity to talk about their own work and interests concerning cultural aspects in learning mathematics. The tone of the meetings will be open and exploratory. Apart from rich sharing, a goal will be to establish directions for future work, and for foci and products that will be appropriate and useful at the meetings of this Working Group in South Africa in 1998.

# DISCUSSION GROUPS

# Discussion Group: Video Tape Interpretation

Laurinda Brown, University of Bristol
David Reid, Memorial University of Newfoundland
Vicki Zack, St. George's School / McGill University

In February we sent some time together in Montréal watching video tapes of Vicki's students[1]. For Vicki this was a usual activity. For David watching video was usual, but the students were younger than he was used to. For Laurinda watching video as a method of research was a new experience. For all of us, watching the videos together was a transformative experience. We discussed wanting to get together again, and wanting to involve others.

The discussion group will watch and discuss one short segment of video data. It will be taken from the wealth of video-taped data of classroom practice which has been generated by Vicki's research project "How is mathematical meaning shared? An investigation of explanations by Grade 5 children during small group problem solving discussions."[2] The intent is to provide an opportunity for discussion of the many issues related to teaching and learning which the tapes might suggest, as well as for meta-discussion of approaches to viewing video tapes as data. On the first day we will concentrate on individual's interpretations of the tape. On the second day we will discuss these interpretations in terms of the different approaches which can be used to analyze video data.

A participant list will be provided to all, and participants will be urged to continue the discussion after the sessions.

We see the relevance of this discussion group to PME in terms of the exchange of methods for research, the centrality of students' activity to education, and the insight into students' thinking afforded by close consideration of their interactions.

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[1] We would like to thank our institutons and the British Council for making this possible

[2] This work is carried out under a research grant (410-94-1627) from the Social Sciences and Humanities Research Council of Canada

# Number Theory Discussion Group

*Stephen Campbell and Rina Zazkis, Simon Fraser University*

Number Theory, the discipline Gauss, Prince of Mathematicians, referred to as the Queen of Mathematics, seems somehow to have been banished from the schools. Exiled to institutes of higher learning, only snippets of the subject are to be found between arithmetic and algebra within the K-12 and preservice teacher curriculum. Perhaps this accounts for why students' learning and understanding of number theory has not been a major focus of research in mathematics education in recent years. We feel the time has come to redress this issue.

Over the past four years we have been investigating preservice teachers' understandings of introductory topics from elementary number theory such as divisibility, prime decomposition, and the division algorithm. The concept of divisibility was one of the first aspects of number theory we investigated in this research. As one of our interview questions, we presented a number $M = 3^3 \times 5^2 \times 7$ and asked participants if it was divisible by 7, 15, 2, 11, 14, 63, etc. Basic questions such as these helped to reveal a wide diversity in students' understandings of prime and compound factors, non-divisors, and of connections between divisibility, division and multiplication. We observed, for instance, that to infer divisibility of M by 7 or 15 was a much easier task for most students, though not trivial for some, than to deny divisibility of M by 11. We have learned much from those participating in our investigations about ways in which they understand these and other related problems. However, the more we have learned, the more we have appreciated the complexities of the domain and the spectrum of students' schemas for the interrelations of the concepts and procedures involved in understanding it. We feel we have only scratched the surface of learners' understanding of number theory and that there is much yet to be studied.

We wish to focus discussion around a few main issues and related topics including:

- the presence and absence of number theory in the mathematics curriculum and:
    - it's relevance in the learning of arithmetic and algebra
    - it's place in the history of mathematics and it's emerging practical application with the advent of computer technology
- preliminary research in learners' understanding of number theory and:
    - the role of number theory in the development of conceptual understanding
    - how we might improve upon our understanding of students' learning of it
- possible directions and methods for further research and collaboration in this area

# Under-represented Countries in PME: Towards the Analysis of Mathematics Education Research Communities

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*"una empresa docente"*
*Colombia*

*Bernadette Denys*
*Paris IV*
*France*

*Susan Leung*
*National Chiayi Teachers College*
*Taiwan*

This group has informally met since PME 18. As stated in the last meeting at PME 20, there are many countries in the world that are under-represented in PME. As an international community, PME should look for a broader international participation in its membership and conferences. This is why it is important to think and discuss about this situation.

There might may be several reasons why many countries are not properly represented in PME. Among them, the following four reasons can be identified:

- Financial difficulties to attend the meetings.
- Lack of information about the PME activities and functioning.
- Deficient competence to communicate in English, PME's official language.
- Lack of research activities that meet the criteria to be presented at PME.

The first three reasons embody problems which are common to most countries and researchers. They should, in principle, be solved if some efforts are made. In fact, the members of the discussion group have already been inquiring about possible solutions. But it is the fourth reason mentioned above which poses many questions and requires an space for deliberation within PME. Unless there is a research community, it is very difficult for a country to be represented in PME.

Let the System of Mathematics Education (SME) in a country be the set of elements (people, institutions, traditions, etc.) and relationships among them which have any influence in the teaching and learning of mathematics and in the research activities concerning this matter. One can then ask several questions concerning the SME:

- How did the research communities of well-represented countries appeared and evolved within their SME's?
- Which is the current state of the SME's of under-represented countries in relation to their mathematics education research communities?
- Which are the key factors that can empower a country's SME so that its research community can develop in an efficient way?
- What role can PME play in this empowerment process?

Based on these questions above, we propose that the discussion in the group meetings at PME 21 be devoted to:

- share the progresses that during 1996 and the beginning of 1997 the members of the group were able to do in respect to the tasks they volunteered to undertake, for analyzing the advances related to the first three reasons, and
- begin the discussion about the four questions posed above, in order to find ways to develop further research on them.

## DISCUSSION GROUP PROPOSAL

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## OPEN-ENDED TASKS AND ASSESSING MATHEMATICAL THINKING

There have been discussions at PME and elsewhere for some time on the use of open-ended tasks for stimulating rich mathematical investigations for students, and for engaging students in processes of mathematical problem solving. There is now considerable attention to the use of such open-ended tasks to collect rich assessment information. Open-ended tasks are being used in tests which are designed to evaluate achievement and also by teachers who are seeking to assess the learning of the students to inform their planning and teaching.

There are a number of problematic issues which arise from the use of open-ended questions for assessment. These predominantly relate to the interpretation of responses, but also to question structure. Some of the issues which can be considered are whether the data can be objectively interpreted, whether rubrics can be constructed which realistically anticipate the range of possible responses, whether particular students are disadvantaged by such tasks, what type of thinking is revealed from the responses, and how are the teachers' interpretation of the open-ended responses to be reported to the students.

This discussion group will consider perspectives on the use of open-ended tasks and share experiences of participants who have been using open-ended questions as the part of assessment. The group will examine what type of thinking is accessed by open-ended tasks and questions and whether open-ended questions provide a way of accessing different styles of thinking than are normally available through conventional closed assessment items.

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# Semiotic Theory and Practice in Mathematics Education

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A review of recent journals and conference proceedings indicates an increasing interest in semiotic theory within the mathematics education community. Research is emerging from a variety of perspectives with foci on signs, symbols, communication and meaning-making all of which may come under the broad banner of semiotics. It is clear that by presenting a point of view of meaning-making that centres on signification and communication semiotics has potential to inform the building of interpretations about mathematics learning that involve both a social and individual dimension, whatever approach is taken.

It may be suggested that, currently within mathematics education, there are three main approaches to semiotic inquiry being invoked. One approach centres around a Vygotskian perspective, another the structuralist point of view, and the third through the work of C.S.Peirce. Although historically each of these perspectives arose independently from within different disciplines they are built on the same basic semiotic assumption, that human beings organise, structure and make meaning for the world in which they live through the use of signs. In particular one may assume for mathematics that activity and meaning-making is only possible through the action of signs. This will be the starting point for this discussion group.

The aim of this discussion group is to bring together ideas and research from a number of semiotic perspectives in order to identify similarities and differences, possible areas of research and development, theoretical notions and empirical results in an attempt to begin to map out the course of semiotic inquiry in mathematics education. We would hope to establish a network of interested researchers throughout the community and continue to exchange papers, ideas and results over the year. It is clearly important that as researchers in an emergent methodology we do not continue to work in isolation.

Initial plans for the meetings include the formation of a network of interested researchers and the sharing of ideas and results. Particular topics of discussion may include: semiotic theories, semiotics as a qualitative research methodology, making interpretations (*i.e.* reading the signs), the value of a semiotic perspective in mathematics education, the semiotics of mathematics *etc.*. Much will depend on the makeup of the discussion group participants. Any one with any interest in the application of semiotics in mathematics education is welcome to attend and contribute.

# SHORT ORAL COMMUNICATIONS

# THE IMPLEMENTATION OF A MICROCURRICULUM: ANGLES, MEASUREMENTS AND ROTATIONS FROM THE POINT OF VIEW OF VAN HIELE. EPISTEMOLOGICAL PROBLEMS

**Afonso Martín, Mª Candelaria, Camacho Machín, Matías. Socas Robayna, Martín**

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The difficulties associated with the implementation of a new curriculum are diverse by nature. Some are of an extra-curriculum nature that have to do with economical, social, syndical aspects. Others are of macro-curricular origin which affect educational institutions, teachers, and students, and others are of micro-curricular source which affect contents and its organisation.

The researches on Geometry from the point of view of Van Hiele, have stressed more in a structure and an organisation of the contents and in a better understanding of the knowledge an behaviour of the students than in the problems that arise when a curriculum is implemented from the teacher's point of view.

The study that we present forms a part of a greater research (Afonso, Camacho y Socas, 1995) in which difficulties and potentialities are analyzed, which are given in the implementation of a Geometry curriculum based on the theory of Van Hiele in Primary and Secondary Education. Changes in the teacher's conceptions and beliefs are analyzed which arise from an immersion study carried out in the development of a didactic sequence on angles, measurements, and rotations, according to the theory of Van Hiele.

15 work sessions (5, 5 and 5) are carried out with a total of 6 teachers. The information taking was the following: video recorded interviews of the teachers' performance in a normal Geometry class; close protocol on the frames of minds of the teachers in relation to Geometry and teaching of it, before and after the experience; control of the teachers' production in the development of the didactic sequence and control of the geometry knowledge by an out-of-the classroom observer as well as of the frames of minds arisen during the didactic sequence.

Through the established category analysis a change can be observed in some epistemological aspects, centred mainly on the teachers' states of minds. In a second phase, their participation in the class will be analyzed, developing the same sequence, with the objective of contrasting with the results attained in the first part of the study.

## References

AFONSO, M.C.; CAMACHO, M. y SOCAS, M.M. (1995). *Some difficulties in the development of the geometry curriculum according to Van Hiele.* Proceedings of the 19th International Conference for the Psychology of Mathematics Education. Brasil.

# STUDENTS' SELF-ASSESSMENT THROUGH JOURNAL ENTRIES

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Making of journal entries does not only get students to actively participate in learning, it also helps them to monitor their learning through self-assessment (Kenney & Silver, 1993). To investigate what sense students made of the mathematics concepts they *learned*, after several *physical* and *mental* class activities which included worksheets covering some of the mathematical concepts *taught*, quizzes, some mathematical investigations, development of some concepts through the solution of problems, and students' group or individual presentations of their solutions to some mathematical problems, I asked eleven preservice mathematics students in Singapore to make weekly journal entries by responding specifically to i) summarise the main points of the lesson, ii) state and explain what you understood from the day's lesson, iii) identify what you did not still understand, and iv) pass any other comments. All the entries were graded.

Results showed that students were very reflective while making the journal entries. More important was the amount of information they provided me. For example, letting students summarise the main points of the lesson gave me a sense of what they considered to be important (most of which I considered important also). Stating and explaining what they understood helped me to identify any misconceptions. Being able to identify what they did not understand was itself a useful learning activity for the students and provided me information on what to re-teach. Finally, they used the opportunity to say anything from cognitive to affective issues to communicate with me the dilemmas they grappled with within mathematics and in their personal lives.

Providing grades for the entries conveyed to the students that their comments were valued. A lot of work was involved in reading, grading, and using information from the entries to plan subsequent instruction. Nevertheless, the benefits in terms of the empowerment of students to take control of their own learning were worth the efforts.

## Reference

Kenny, P. A. & Silver, E. A. (1993). Student self-assessment in mathematics. In Norman L. Webb and Authur F. Coxford (Eds.), *Assessment in the Mathematics Classroom 1993 Yearbook, 229-238*. Reston, VA: National Council of Teachers of Mathematics.

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# 'COMPUTER CONCEPTS' OF MATHEMATICS AND COMPUTER SCIENCE TEACHERS

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**ABSTRACT.** *All over Europe, great efforts have been made to introduce computers and new media to schools. Teacher education and in-service-trainings mainly focus on cognitive aspects, attaching importance to imparting up-to-date technical knowledge. However, my study has revealed that especially affective factors have a major impact on the teachers' performance. In addition, the study demonstrates a change of teaching paradigms from mathematics to computer science classes in German schools.*

The aim of the study was to investigate the hypothesis that, by analogy with *mathematical beliefs*, there are also specific *computer science beliefs* to be found among mathematics and computer science teachers. The work is mainly based on qualitative methods, with the empirical material deriving from 30 in-depth video-taped interviews (1.5 to 2 hours each) with teachers from German grammar and comprehensive schools.

It is widely thought that the effective teaching of computer science is predominantly influenced by the teachers' cognitive skills, with emphasis on up-to-date technical knowledge. However, as the study has revealed, the individual *'computer concept'* of a teacher, i.e. a complex structure with cognitive, affective, and operational components, considerably influences the teaching process. It plays the part of a personal 'hidden curriculum', having a selective and directive impact on the teacher's performance.

There are mainly three fields or scopes of experience forming a teacher's computer concept, i.e. science, school, and society (or everyday life) – with three corresponding social roles, namely those of an expert (of mathematics and computer science), of a teacher, and of a private person. From 'science' via 'school' to 'society', the computer views of each interview partner take a more and more emotionally charged perspective, while the role of the computer is increasingly considered as central and relevant. The more the theoretically specialized character of the field declines and social aspects and everyday experiences become determining, the greater the importance (approving or disapproving) attached to the role of computers becomes. Not the view of the 'expert' is computer-centred, but the view of the 'private person' is.

The interviews manifest general differences in the teachers' views on mathematics and computer science. Whereas maths is seen as oriented towards theory and formalism, and maths teaching as frontal, teacher-centred, and dogmatical, computer science is seen as practical, concrete, interdisciplinary, and oriented towards applications and problem solving. Teaching computer science is associated with keywords such as teamwork and creativity. While teachers in maths classes mostly stick to the traditional teaching paradigm (keywords: lesson, homework, classroom test, teaching, examining, educating etc.), in computer science classes there is evidence of a *change of teaching paradigms* towards an innovative professional life model with the leading concepts of project, product, team, discussion, consulting, delegating, and co-operating.

# DESIGN PRINCIPLES FOR DEVELOPING INTEGRATED MULTIMEDIA INSTRUCTIONAL MATERIALS

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Kay McClain, Paul Cobb
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This paper describes the design process that guided the development of a CD-ROM environment created for prospective mathematics teachers. The goal of the CD-ROM (Goldman, et al., 1994) was to develop a shared context to support discussions about critical pedagogical and content issues that can arise in reform-based classrooms. To this end, our first step was to plan a sequence of three geometry lessons. This development involved anticipating 1) various ways in which the students might interpret the proposed activities based on prior research, and 2) how the teacher might be able to build on students' mathematical contributions while achieving her pedagogical agenda. Once the lessons were filmed, we reflected on the critical incidents in the video in order to design the CD-ROM activities for prospective teachers. Once again, we anticipated the various ways in which our preservice students might interpret the activities and how we, as instructors, might build on our students' contributions while at the same time highlighting the incidents that we felt addressed our pedagogical agenda.

The parallels between the two development efforts illustrate our view of the reflexive relationship between development and research, which is consistent with the Freudenthal Institute's program of Realistic Mathematics Education (RME). The critical difference between these approaches and that of traditional instructional design is that we attempt to anticipate students' interpretations rather model the ways in which experts might solve the tasks. This bottom-up approach (Gravemeijer, 1994) differs from the top-down model of first wave instructional design in which a task analysis is conducted in order to determine a means-end rationality. For example, the teacher in the geometry lessons does begin the class by showing examples of how to use the formula for calculating volume. Instead, she attempts to support the students' own ways of acting with the materials to develop deeper understandings of volume in terms of layers. Similarly, we do not view the CD-ROM as a means for showing "exemplary teaching." Instead, we view it as means for supporting teachers' own efforts to reorganize their beliefs and practices as they attempt to understand the development of students' thinking and the teacher's role in guiding it.

Goldman, E., Barron, L., Bassler, O., Cobb, P., Bowers, J., McClain, K., Robinson, C., St. Clair, J., Harwood, J., & Altman, J. (1994). *Investigations in Teaching Geometry* [Computer program]. Nashville, TN: Vanderbilt University. (Produced in association with National Science Foundation Grants No. TPE-8751472, TPE-8950310, TPE-9053826, and TPE-9154067)

Gravemeijer, K. (1994). Developing Realistic Mathematics Education. Utrecht, The Netherlands: Utrecht CD-ß Press.

# Teachers conceptions of the domain of mathematics and its effect on teaching in primary classrooms.

Mary Briggs
Mathematics Research Centre, University of Warwick

This exploratory study focuses on the differences in approaches to planning and teaching mathematics in the primary classroom and other subjects in the primary curriculum. The methodology has been to use a combination of observation and interview. Teachers have been observed teaching mathematics and other subjects in early years classes in order to make comparisons of organisational and teaching strategies across subjects.

In interviews the notion of 'domain' is explored and linked to views of teaching and organisation across a range of subjects within the primary curriculum. Patricia Alexander (1992) offers a simple definition of domain as 'an area of study'. This might imply that this is unproblematic yet it is a highly contentious issue.
The domain of mathematics is seen in specific ways by most primary teachers, often quite restrictive in terms of the range of teaching strategies for mathematics. Yet other subjects are viewed as featuring very different 'domains' with some linking to other subjects naturally. The same teachers teaching other subjects are seen to be creative and flexible in their teaching strategies and organisation.

The organisation and planning of teaching of mathematics to lower primary pupils is influenced by the availability and use of published scheme materials, teachers competence and confidence with mathematics and mathematics teaching. Though the overarching issue is their conception of mathematics as a subject domain.

ALEXANDER, P.A. (1992) 'Domain knowledge: evolving themes and emerging concerns' *Educational Psychologist*. **27** (1) 33-51

# CONCEPT FORMATION AND REPRESENTATION: A STUDY ABOUT TRIANGLES IN FIRST GRADE STUDENTS

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The purpose of this study was to investigate, based in a cognitive point of view, the level of concept formation and the representation of triangles in first grade students. The study is based on Klausmeier's (1974) model of concept formation which proposed four levels of concept development (concrete, identity, classification and formal) and on Van Hiele's model of geometric thinking (Burger and Shaughnessy, 1986, Pegg and Davey, 1991, Matos, 1992, Nasser, Zapata and Bernardo, 1996).

The developmental levels proposed by Klausmeier are qualitatively different from one to the other and they have a structural organization. The students develop different kinds of cognitive operations at each level and the subsequent level is more sophisticated than the preceeding one. Anderson (1995) notes that complex mental images are organized into pieces and formed from a hierarchy of images. The learning of a chunk (unit of knowledge representation) may facilitate the learning of more complex mental images.

The subjects (32 students of a private school) were asked to draw examples of triangles and to identify the concept of triangles in terms of their defining attributes and in terms of examples and non-examples. The material used to collect the data was a questionnaire, a test about the defining attributes of triangles and a test about examples and non-examples of triangles, used and analized in a previous master's thesis (Pirolla, 1995).

The data was analized in a comparison between cognitive operations at all levels and the results obtained by the student on the tests and when he/she defines triangle and draw the figure. The results show that students have the concept of triangle on a concrete level. Just few students present the cognitive operations necessary for more complex levels. The majority of students present difficulties in working the concept on a classification level, because at this level the student is supposed to recognize all types of triangles and include them in a broad class, while at the same time, being able to re-organize them in each particular class. The presentation will provide some examples of protocols with the students' drawing.

**References:**

Anderson.J.A (1995) - Cognitive Psychology and its Implications. 4th ed. New York: W.H.freeman and Company.

Burger,F. and Shaughnessy. (1986) - *Caracterizing the Van Hiele levels of development in Geometry.* Journal for Research in Mathematics Education. 17 (1): 31-48.

Klausmeier, H.J (1977) - *Educational experience and cognitive development.* Educational Psychologist. 12(2): 179-196.

Matos, J.R (1992) - *Acomodando a teoria de Van Hiele a modelos cognitivos idealizados.* Quadrante. Lisboa (1): 93-112.

Nasser, L., Zapata, G.O.G. and Bernardo, L.C.M. (1996) - *An expert system to guide the geometry teacher.* Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education. Eds. Luis Puig and Angel Gutierrez. Vol I: 1-188.

Pirolla N.A (1995) - Um estudo sobre a formação dos conceitos de triangulos e paralelogramo em alunos de 1° grau. Dissertação de Mestrado. Universidade Estadual de Campinas.

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# INTERACTIVE MULTIMEDIA AND PROBLEM SOLVING ON PROPORTIONALITY

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Research in the domain of the acquisition of mathematics concepts has shown that one of the critical aspects of that knowledge lies in the resolution of problems of proportionality.

Interested both in a better understanding of the reasons underlying the difficulties detected in the acquisition of the model of proportionality and in trying to find possible ways to aid its construction by the students, a study was carried out in a real classroom context.

The school, for various reasons, leads students, when confronted with exercises, to pay attention solely to coming up with an answer that might be valued, that is, mechanically apply the algorithm previously presented by the teacher, without placing any effort into understanding the underlying proportionality concept.

Given the advantages that interactive programmes can introduce in the teaching/learning process, an interactive multimedia programme was developed in which the authors try to privilege the acquisition of the model as to the algorithms of resolution.

This study is still in its developmental stage and in spite of the results already obtained, the aim of this paper will be to discuss the computer programme itself. This discussion aims at contributing towards the perfecting of the programme, and especially: the way in which one tries to conceptualize the model from concrete examples;the way students can solve the set of problems which the programme contains; the relationships between the previous two; the interactive nature of the programme as it pertains the resolution advanced by the student.

# INFLUENCES OF SIGNIFICANT PERSONALITY FACTORS ON STUDENTS MATHEMATICS LEARNING.

A survey 10 years study with 20 young children.

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In this paper, we report on a study carried out from 1980 to 1990 with 20 children aged 1 to 9. It is part of a broader project that aims to find out apparents influences of some factors on student Mathematics learning. The following variables have been taken into consideration: early stimulation, family communication, child's attitudes, chronological age, thinking abilities, reactions and answers to mathematics and Computers problems, out-to-school activities, continuity of some notions learned at school and the child's achievement in Mathematics. During this study we have gotten certain findings about each variable in an independent way and interesting correlations between them. Concerning the sample, children were from families with similar both, intelectual background and socio-economical level; children's mother, should be involved for a tight collaboration. The data collection has been got through direct observation interviews, school work revision, records and school reports cards.There was direct and close relationship with children; recording their behavior and anecdotes once at week. Children's mothers and teachers, were interviewed at least everv six months.

Every year the data collected were summarized in an information chart for allowing children clasification according the presented characteristics.

Piaget's theories and constructivist perspectives were taken as a basis for the theoretical framework.

We want to highlight as a conclusion the need to encourage the Mathematics learning in such environment where family relations, school and society were tightly integrated.

**References:**

CAMPOS CAMPOS Yolanda y Eloísa BERISTÁIN MÁRQUEZ. *Matemática y realidad. Propuesta de una didáctica integradora para la Educación Básica.* Mexico: ANPM, 1983.
MAKARENKO, Antón S. *El poema pedagógico.* Russia: MIR, s/a. 235 p.
PIAGET, Jean. *¿A dónde va la educación?.* Mexico: Teide, 1983. 110 p.
PIAGET, Jean. *La toma de conciencia.* Spain: Ediciones Morata, 1980. 283 p.
PIAGET, Jean y Noam CHOMSKY. *Teorías del lenguaje, teorías del aprendizaje.* Spain: Ed. Crítica, 1984. 389 p.

# Graphic calculators and problem solving. Do they help?

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Current research on the use and effects of graphics calculators shows that their use can enhance the learning of functions and graphing concepts and the development of spatial visualization skills. They can also promote a shift from symbolic manipulation to the graphical investigation and examination of the connections among the several representation systems associated to a given concept (Penglase and Arnold, 1996). However, few studies explore the way graphic calculators are used by students while solving problems and the relationship with, and the effects its use might have on, the understanding of specific concepts. In this study we analyzed the performance of a group of students in a precalculus problem solving situation in which they were allowed to use graphic calculators.

We were interested in looking at the role the graphic calculator could play in the appearance of "disturbances" (identified as situations in which there was a difference between what the students expected and what they found while solving a problem) and the way its use could promote the consolidation of previous knowledge or the construction of new knowledge in order to solve this difference. A one hour problem solving activity concerning the analysis of transformations of functions in the graphical and symbolic representations with three students using graphic calculators was videotaped. The interaction among the students was transcribed. A situation in which a disturbance appeared was identified. The way the students faced this situation was analyzed from a microethnographic point of view (Voigt,1989).

We found that, while facing a disturbance, students might use graphic calculators to consolidate previous and partial knowledge. The authority role given by the students to the information provided by the calculator, their deficient use of this information, and the restricted ways in which they read and interpret graphs (consequence of previous teaching without calculators), might induce them to misuse this information, driving them to ignore alternative situations in which new knowledge could be constructed. Furthermore, we found that graphic calculators might induce students to work almost exclusively in the graphic representation system neglecting the verbal and symbolic representation systems.

Even though graphic calculators use might have favorable effects on students' understanding and problem solving skills (measured on the basis of their performance in tests), there might be difficulties involved in this learning process.

**References**

Penglase, M., Arnold, S. (1996). The Graphics Calculator in Mathematics Education: A Critical Review of Recent Research. *Mathematics Education Research Journal*, 8 (1), pp. 58-90.

Voigt, J. (1989). The social constitution of the Mathematics province — A microethnographical study in classroom interaction. *The Quarterly Newsletter of the Laboratory of Comparative Human Cognition*, 11(1 & 2), pp. 27-34.

# PUPILS' INTERACTIONS IN MATHEMATICAL TASKS

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The processes of knowledge aprehension and skills acquisition are quite complex and they are influenced by many psychological and social factors. Since the first introductory studies by Doise, Mugny and Perret-Clermont (1975, 1976) that social interaction has been pointed out as one of the main facilitator factors for pupils' performances and for the implementation of their cognitive and learning progresses. Peer interaction is one of the modalities of social interaction that has been more extensively studied in the last decades and which has proved to be an important factor for subjects' acquisition of knowledge and skills.

In this paper we will discuss the role of peer interaction in the learning process of mathematical knoewledge. We asked 7th grade pupils (98 dyades), attending a school in the surroundings of Lisbon and integrating an innovation project, to answer to a "non-usual" task (indicated by teachers as not traditional, nor typical), related to the balance metaphor (equations). Peer interaction was taped and we will use part of these protocols to illustratate the role played by peer interaction in the choice of solving strategies and in the level of performance of the subjects.

A deep analysis of this kind of protocols is essential to understand the role that peer interaction may have in the acquisition process of knowledge and skills and it stresses the positive effect of peer interaction in development and in school achievement.

References:

Brun, J. and Conne, F. (1990). Analyses didactiques de protocoles d'observation du deroulement de situations. *Education et Recherche*, 12, 3, 261-286.

César, M. (1994). Factores psico-sociais e equações. *Actas do ProfMat 94*. Lisboa: Associação de Professores de Matemática, 82-92.

César, M. (1995). Interacção entre pares e resolução de tarefas matemáticas. *Actas do VI Encontro de Investigação em Educação Matemática*. Lisboa: Associação de Professores de Matemática.

César, M. (1997). *O papel da interacção entre pares na resolução de tarefas matemáticas*. Col. Horizontes pedagógicos, nº 23. Lisboa: Instituto Piaget. (in press)

Doise, W., Mugny, G. and Perret-Clermont, A. N. (1975). Social interaction and the development of cognitive operations. European Journal of Social Psychology, 5, 3, 367-383.

Doise, W., Mugny, G. and Perret-Clermont, A. N.(1976). Social interaction and cognitive development: further evidence. European Journal of Social Psychology,6,2,245-247.

Perret-Clermont, A. N. and Nicolet, M. (1988). Interagir et Connaître - Enjeux et régulations sociales dans le développement cognitif. Fribourg: DelVal.

Schubauer-Leoni, M. L. and Perret-Clermont, A.-N. (1980). Interactions sociales et représentations symboliques dans le cadre de problèmes additiffs. Recherches en Didactiques des Mathématiques, 1, 3, 297-350.

Schubauer-Leoni, M. L. and Perret-Clermont, A. N. (1985). Interactions sociales dans l'apprentissage de connaissances mathématiques chez l'enfant. In G. Mugny (Ed.), Psychologie sociale du développement cognitif. Berna: Peter Lang, cap. 11, 225-250.

# MONOTONUS MATHEMATICS LESSONS AT THE PRIMARY LEVEL: THE BEGINNING OF AN END

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Although the main study is built on a tested causal model, this report is based on the data gathered from 7 out of 18 teachers who were observed. This paper looks at the predominant teacher behaviours and student attitude to math.

Using an observation instrument, four lessons of each of the 7 teachers were observed and coded. The instrument gathered information on the initiator of interactions, to whom it is directed and the nature of interactions. Student attitude was obtained by giving a list of school subjects where they wrote one or zero for like or dislike as well as the reasons for their views.

Although 28 lessons of 7 teachers with experiences ranging from 10 to 25 years were observed, the pattern of interactions were more or less the same. Lessons were dominated by teacher explaining the content or procedures, chalk - talk, directives to the students, recall questions and procedures like copying questions from the text book to the black board. The one and only student activity was oral or written response to the teacher.

Most of the lessons (75%) were review hence repititive. It is no wonder that one grade of four classes indicated dislike for math at pre observation averaging 38% which jumped to 57% at the post observation, while the other grade had 45% for both pre and post observation.

For teachers to be innovative and reflective in their teaching, the training procedures need to be updated. Although reform in education is taking momentum, immediate solution to the problem is short inservice to update both subject knowledge and teaching skills otherwise more and more students will become 'anti mathematics'.

References:

Chacko, I.(1989) The relationship between selected teachers' verbal behaviour and students' achievement in mathematics. Int. Jnl.of Math Educ.in Sciec. and Technology. 20:1, 63-71.

Van der Sijde Peter C. (1987-88) Relationships of classroom climate with student learning outcomes and school climate. 23:2, 40-44

# DEDUCTIVE PROOF: A GENDER STUDY

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Currently there is a renewed interest in the learning and teaching of mathematical proof (Hanna, 1996). Traditionally secondary school students were introduced to mathematical proof through Euclidean geometry. However, due to a formalistic approach to the teaching of geometry, Euclidean geometry fell into disfavour in most western countries. Attempts to replace Euclidean geometry, as an introduction to mathematical proof at secondary level, appear to have failed. Consequently students are insufficiently prepared to handle mathematical proof in tertiary studies.

Because Euclidean Geometry is a compulsory part of mathematics for at least three years at the senior secondary level in South Africa, research on the writing of deductive proof is still done in South Africa (de Villiers, 1990). The research reported on here is part of a gender study. Research findings show that, if gender differences in mathematics performance exist, it is more likely to be found in the field of spatial ability and from adolescence onwards.

An experiment was conducted with a large sample of grade 11 students from 5 different high schools. The test instrument, a variety of geometry problems, was part of the students' end of the year examination. The geometry content was of a Euclidean nature and included analysis, synthesis, deductive reasoning, hypothesis testing and proof writing. A fair ability at proof writing was required to obtain a good mark.

A detailed diagnostic evaluation of all students' responses was done by the researcher, taking special note of differential gender performance on important aspects like:

logical geometrical reasoning
rote memorization
proof writing

Though females showed more evidence of rote memorization than males, **no** marked gender differences were found in the diagnostic evaluation. The two gender groups performed equally well on proof writing in Euclidean Geometry.

De VILLIERS, M.D. (1990) The role and function of proof in mathematics. *Pythagoras* 24, 17-23.
HANNA, G. (1996) The Ongoing value of Proof. In L. Puig & A. Gutiérrez (eds) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education.* Valencia, University of Valencia.

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# THE INTEGRATION OF TI92 AT HIGH SCHOOL LEVEL : A BIOGRAPHICAL APPROACH

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Since December 1995, we have been involved in a research project dealing with the integration of the calculator TI92 at high school level (16-17 years old students). The purpose of this research was to determine and to analyse:

* the processes underlying the instrumentation of the TI92.
* the mathematical knowledge required for this instrumentation.
* the articulation between this kind of knowledge and the one expected by the institution.

The data collected includes questionnaires, regular classroom observations, classroom assesments and a biographical study of 9 students selected according to their sex, math level and relationship with technology.

In the oral communication, after presenting the research project we will focus on the results obtained through biographical approach (especially through regular interviews). We use it with the aim of analyzing the strategies developed by the students in order to study functions and their evolution along the academic year.

Our analysis is based on the notions of " instrumented techniques " and " schemes ". We will try to elucidate some complex phenomena underlying the construction and evolution of these " instrumented techniques " as well as their relationship with their standard paper/pencil analogues.

## References

Artigue, M., Defouad, B. & Lagrange, J.B. (1996) *Intégration de calculatrices complexes dans l'enseignement des mathématiques au lycée*. IREM, Université Paris 7.

Rabardel, P. (1995) *Les hommes et les technologies - Approche cognitive des instruments contemporains*, Armand Colin, Paris.

Trouche, L. (1996) *A propos de l'apprentissage des limites de fonctions dans un "environnement calculatrice" : Etude des rapports entre processus de conceptualisation et processus d'instrumentation.* Thèse de doctorat. Université Montpellier II.

# "ARE THERE DIFFERENT TYPES OF MEASURING NUMBERS?"

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Measuring numbers arise from comparing two scalar quantities (of the same kind): by conceiving the idea that it is always possible to express one quantity as a multiple of another (Cable, 1976). We explore in this paper children's understanding of another type of measuring numbers with different properties and allowing for different operations. We suggest that the above conception only applies to extensive quantities: intensive quantities are measured indirectly, not by reference to a quantity of the same kind. There is a second, significant difference between extensive and intensive quantities: in extensive quantities, the whole is equal to the sum of the parts; in intensive quantities, the part and the whole have the same value. Thus operations with extensive and intensive quantities differ: whereas addition and subtraction can be carried out on extensive quantities, in intensive quantities the corresponding operations are mixing and sampling. We hypothesise that children's well documented difficulties in working with numbers that represent intensive quantities relate to their lack of awareness of the differences between intensive and extensive quantities.

We tested this hypothesis by asking children (9 to 11 years) to indicate the results of mixing and sampling from intensive quantities (colour of paint and concentration of orange juice). The tasks were given either in the presence or in the absence of numbers. In the mixing task, the children were shown six patches of the same hue varying in saturation; the patches were pasted on a card organized from least to most saturated forming a perceptual scale. The children were asked to show on the card what colour would result from mixing varying amounts of paint of the colours shown in the scale. Under each patch there were numbers or words which expressed the value of saturation. Some mixtures were of the same and others of different hues. In the sampling task, the children were presented with a picture of containers of orange juice from which samples were taken. The samples varied in origin (same or different concentrations in the original jars) and size (larger or smaller samples). The children indicated whether the samples tasted the same or not as each other and as the juice in the jar.

Both mixing and sampling were difficult for the children: overall rates of success were about 50%. From the wrong answers, 70% resulted from treating mixing as addition and 90% from treating sampling as subtraction. Children who had succeeded in simpler mixing trials often failed in more complex trials, when different amounts of different hues were used. Younger children performed significantly better when words rather than numbers were used in the mixing task. This suggests that they understand relations between intensive quantities earlier than their quantification but alternative explanations need to be considered. These results suggest the need to consider children's basic ideas about intensive quantities before they are introduced to measuring numbers for intensive quantities.

References
Cable, J. (1976). Who's afraid of the measuring numbers?. *The Mathematical Gazette*, 60, 412, 81-89.

# TREE DIAGRAMS IN PROBABILITY : A REAL REGISTER OF REPRESENTATION .

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These last years, the researches initiated by Raymond Duval in the IREM (Institute of Research on Mathematical Education) in Strasbourg (France) have been concentrated on the study of the role of moving from one register of representation to another in a context of mathematical learning.

The idea of register comes from a semiotic point of view, which means that you study any system of signs allowing people to represent or to communicate. For us, registers are systems of representation which have a structure rich enough to allow "treatments" (internal transformations) and "conversions" (transformations of representations given in one register into representations in another register ).

Everyone doing mathematics uses several registers of representation : natural language, symbolic language, graphics, cross tables, tree diagrams, etc, and moves from one register to another without paying attention to these changes. We consider that this activity of "coordination" of registers is necessary for mathematical learning and must be explicitely taught.

We have studied the initiation to probability from this point of view ; the elementary problems are often a mixture of natural language and symbolic language. One main difficulty for students, even when they know theorems and definitions by heart, is to select in the text of a problem the data they have to operate on. Then the question is : which tool should we teach to students in order to allow them to select the data and find in the text the relationships between them ? The idea is to look for pertinent tools in registers of representation different from natural and symbolic language. These tools may be tree diagrams or crosstables, according to the nature of the problem. We have provided 18 to 20 year old students with different problems of probability and observed significant variations.

DUPUIS C. et ROUSSET-BERT S. (1996) Arbres et tableaux de probabilité : analyse en termes de registres de représentation. *Repères-IREM* 22, 51-72.

DUVAL R. (1995) *Sémiosis et pensée humaine* Bern, Peter Lang.

# LEARNING MATHEMATICS AND LANGUAGE TOGETHER

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The goal of the research was to design and test ways to help native Spanish speakers to learn mathematics while at the same time improving their English language communication skills. The research involved 140 students from ages 10 - 12, most of whom spoke Spanish as a first language. Research on subject matter and language learning involves multiple layers of complexity. For example, learning mathematics in a second language involves not only the first and second natural languages, for example, Spanish and English, but also the mathematics "register," that is, the technical vocabulary specific to doing and talking about mathematics. In this study, students were grouped in heterogeneous small groups which included high and low proficiency speakers of English. In addition, Spanish translations of key terms and problem statements were available to the students. The purpose was to insure that students with limited English proficiency (LEP) would have access to both peer support and written materials to assist their understanding of the mathematics. The groups engaged in 20 minutes of collaborative problem solving for four days a week over a period of four weeks, and also worked on one challenging "Problem of the Week," the latter required a written solution from each student. Students were pre-tested in three areas: mathematical problem-solving; written mathematical communication; and English language reading skills. The short oral presentation will summarize the results of post testing on these same areas, as well as a qualitative analysis of selected problem-solving episodes

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## *LIMITS IN SECONDARY EDUCATION: HOW TO ANALYZE THE TEACHER'S ACTIVITY.*

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We present some methodological aspects of the continuity of our research (Espinoza, Azcárate 1995) that is being developed in the frame of the "anthropological approach to didactics" (Chevallard, 1992). One of its aims is to describe the techniques that the teacher uses in order to direct the process of studying the limits in secondary education. Furthermore, we will show some results that manifest both the pertinence of the instruments constructed underneath this point of view, and their capacity to describe and explain some didactic phenomena.

**Purpose:** To describe the action instruments both conceptual and material, that the secondary teacher uses in his task of organization and management of the learning of his students about the limit concept.

**Methodology: 1.** Two 10th grade teachers are selected and a pursuit of the process of studying relative to the limit object is made. **2.** The *Didactic Moments Theory* is used to analyze this process. The different dimensions which are present in the respective process of studying are identified, and some tables which discretize the process are built according to the dominant *moment* of the activity. **3.** Some contract ruptures are explored, especially those which are manifested when both teacher and students are in different *moments* of the process of studying. And we also show the presence of some epistemological obstacles (Brousseau, 1983).

**Hypothesis:** In order to analyze and describe the teacher's thinking (his conceptions, teaching styles, etc.) we should not take for granted the mathematical activity which is performed by the teacher and the one which is supposed to be performed by the students. This analysis helps us to identify the mathematical and institutional restrictions with which the teacher is confronted and which strongly affect his educational labour.

**Some Results: 1.** There are no coincidences between the characteristics that the teacher attributes to the knowledge to be taught and his concrete performance in the classroom. **2.** In spite of the differences between the two observed teachers leading the process of studying, the students' difficulties are similar.

We will show that the characteristics of the mathematical organization to be taught are more definitive for the student's learning rather than the teacher's personality characteristics or teaching styles. We will conclude that any attempt to describe the teachers or students conceptions related to the learning or teaching process of some mathematical object, should start with the analysis of the restriction imposed by this organization.

**Some References:**

Brousseau G.: 1983, Les obstacles épistémologiques et les problèmes en didactique, *Recherches en Didactique des Mathématiques* 4.2. Grenoble.

Chevallard Y.: 1992, Fundamental concepts in didactics: Perspectives provided by an anthropological approach, *Recherches en Didactique des Mathématiques, Selected Papers*, 131-168.

Espinoza L., Azcárate C.: 1995, A study on the secondary teaching system about the concept of limit. *19th-PME Conference Proceedings*, Brazil, Vol. 2, 11-17.

# Understanding of the notions of p–value and significance level in the solution of hypothesis tests problems

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Shaughnessy (1992) suggests the need to conduct clinical teaching experiments in which the researcher is also the teacher and in which the effect of teaching on the construction of conceptual knowledge is followed. In this line of research, the authors (teachers–researchers) carried out an exploratory study in which they designed, on the basis of a didactic analysis similar to the one proposed by Vallecillos (1996), three problems of hypothesis tests. Our purpose was to analyze the effects of graphic calculators use in the application and understanding of the concepts of p–value and significance level in the solving of these problems. The study was done with students of a statistics course for Social Sciences at the university level.

We studied, on the basis of the problem solving activities done by the students, their errors and difficulties concerning the concepts of significance level and p–value. First, each problem had to be individually solved in one hour, and answers given in a writen form. Then, during the second hour, the students compared their individual solutions in groups of three people, in order to produce a joint solution. The information used in our analysis came from: 1) the written productions of the students (individual and joint); 2) audiotaping of one group´s interaction; 3) interviews with the three students whose interaction was audiotaped.

We found that the graphic calculator was used only slightly in order to represent the p–value. Even though each student "seems to understand" the p–value concept, he/she does not use it for deciding about the hypothesis test. We thought that the graphic calculator use was going to promote the use of the graphic and numeric representations. However, in order to decide about the hypothesis test, the students preferred to compare percentiles, neglecting the comparison between the p–value and the significance level.

This result and other similar ones suggest the need to reflect about the phenomena that put into play the concepts in question, the kind of didactical activities that are designed and used in order to work with these concepts, and the conceptions and obstacles which are behind the errors made by the students.

## References

Shaughnessy, J. M. (1992). Research in probability and statistics: reflections and directions. In D.A.Grouws (Ed.). *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan, pp. 465-494.

Vallecillos, A. (1996). *Inferencia estadística y enseñanza: un análisis didáctico del contraste de hipótesis estadísticas*. Granada: Comares.

Title: Communicating Mathematical Ideas: Reflecting and Convincing

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The purpose of this study is to better understand how students develop awareness of mathematization. This report is a component of a cross-cultural, longitudinal research that is being conducted by mathematics educators from Universidade Santa Úrsula and Rutgers University-Newark.

Four third grade students (9 years old) were videotaped while building towers with unifix cubes. This study took place in a private school in Rio de Janeiro, Brazil, in 1996. Theoretical foundations are based on Gattegno's model of awareness, Powell's model of mathematization, and Frant and Bairral's model of didadatical relations. The proposed task was to build 4-high towers, selecting from two colors of unifix cubes. This task is based on the longitudinal research that has been conducted by Davis and Maher (1995).

Once the students felt that they had constructed all possible towers 4-high, the researchers challenged them to explain how they knew that they had all the possibilities. The questions posed were intended not to lead the students thinking but rather to have students expose their awareness of their strategies. Studying the videotapes, allowed us to observe several interesting aspects of how the students organized their thinking. As students explained their strategies, the researchers noticed that the students talk contained two distinct characteristics. At first, some students communication had the character of "talking aloud," organizing their own reflections and understanding deeply their strategies. At another moment, students communication changed from "talking aloud" to "convincing others." Here students are confident about their strategies and wish to have others affirm their statements.

We were able to identify some of the organizing strategies students used for constructing and counting the number of possible towers 4-high. Besides addressing research into students mathematical thinking, we will discuss implications of our work for teacher education. The communication issues raises the following questions. To what extent do mathematics teachers, provide an environment for students to think aloud and refine their thoughts while communicating to others? How can we use manipulatives in classrooms not only as a metaphor but also as an object for thinking?

References

Davis and Maher 1995 How students think. The role of representation (copy)

Frant and Bairral in Buscando semelhança encontramos mais do que meras coincidências. Unpublished Master's Dissertation. Universidade Santa Ursula, RJ

Gattegno, C. (1988). The science of education. Part 2B. The awareness of mathematization. New York: Educational Solutions.

Powell, A. B. (in press). Awareness of Mathematization

# THE CONCEPTUAL FIELD OF DISCRETE ADDITIVE OPERATORS: RELATIVE NATURAL NUMBERS AND MEASUREMENTS

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The present report displays a summary of the main results of a research aimed to clarify and organize a part of the additive conceptual field (Vergnaud, G., 1993: 97) which is called *Conceptual field of relative natural numbers* (González, J. L., 1995). The study has been carried out in two phases of development: theoretical and empirical. The theoretical study, developed by means of the "didactic analysis"[1] on the domain of the phenomena and problems relating the practical application of the natural numbers and integers, provide arguments proving the existence of a subdomain involving metrical and numerical notions - relative natural numbers and measurements - related to the interpretation of integers as additive operators (Vergnaud, G. et Durand, C., 1976) originated from the comparison of natural numbers and measurements. Nevertheless, these notions, which are usually referred to as integers, operate with some properties showing five differences in relation to the ordered additive group of integers: 1) Total order / partial order with inversion at "negative region"; 2) Without first element / with first element; 3) Connection / disconnection between dual regions; 4) Unique zero / double zero; 5) Integer addition / additive annulment - compensation (relative natural addition). The empirical study, for which four questionnaires have been used in order to analyse the semantic differences between both kinds of concepts, shows the existence of some cognitive differences. Endowing the identified field with an own entity makes the results of other researches clear, gives an answer to several didactic questions, establishes a new distribution for the additive field and a new classification of the additive word problems.

## References

**González, J. L.** 1995. *El campo conceptual de los números naturales relativos.* Tesis Doctoral Universidad de Granada. Microfilm: SPICUM Univ. de Málaga.

**Vergnaud, G. et Durand, C.** 1976. "Structures Additives et Complexite Psychogenetique". *Revue Française de Pedagogie* 36: 28-43.

**Vergnaud, G.** 1993. "La Teoría de los Campos Conceptuales". Sánchez, E. y Zubieta, G. eds. *Lecturas en Didáctica de las Matemáticas. Escuela Francesa.* Cinvestav -IPN. México DF. 88-117.

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[1] Methodological procedure which relates and integrates data coming from the four basic information sources: History and Epistemology, Learning and Cognition, Phenomenology and Teaching and Curricular Studies, by following a sequential process according to the qualitative meta-analysis criteria (González, J. L., 1995: 58-62).

# THE IMPACT OF IN SERVICE MATHEMATICS WORKSHOPS

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"South African pupils performed worst in both mathematics and science in an international study conducted among Std 5 and Std 6 pupils in 41 countries." IEA TIMSS-SA, 26th November Media Release from the Human Sciences Research Council (HSRC).

The Mathematics High schools In service Project (MHIP) was formed in February 1996 at the RADMASTE Centre. It was started as a response to research which shows that the majority of secondary school mathematics teachers are inadequately prepared to teach mathematics. It runs workshops with teachers in clusters of schools in and around Johannesburg. Workshops are highly interactive and deal with both content and methodological issues. They run over a period of 3 - 6 months. Regular support visits and classroom observations enable good working relations to develop between teachers and facilitators/ researchers.

The research work that is discussed in this paper was conducted in 20 schools in traditionally neglected and under resourced areas around Johannesburg. 25 grade 7, 8 and 9 mathematics teachers attended the workshops. The aim of the research has been to ascertain the present situation in schools in which the MHIP has been involved and the impact the project has on the teachers in these schools. This paper will take a brief look at:

- The research design

Data was collected in the form of pre and post questionnaires, observations and interviews and processed in three main areas namely: teachers' beliefs, attitudes and experiences of mathematics education, pupils' beliefs, attitudes and experiences of mathematics education, and the testing of pupil knowledge at grade 7 level.

- The results of the testing of pupil knowledge at grade 7 level

A mathematics test was given to 188 grade 7 pupils. The test was designed in order to analyse and compare pupil performance on process type questions versus straight forward algorithmic type questions and on context based questions versus 'pure' maths questions. Analysis of results yielded some interesting findings and raised useful questions for future research. Clearly students performed worse on process based questions and the placing of questions in a familiar context had a mixed effect on pupil performance.

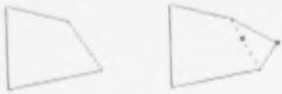
- A brief summary of other research findings

Minor changes occurred in teacher conceptions of mathematics teaching. Pupil conceptions however remained consistent. Teachers were clearly working within the traditional framework of teacher centred, rule based teaching. A serious limitation of the research has been the limited time in which the research was conducted.

# AN ATTEMPT TO CHARACTERIZE ENVIRONMENTS IN WHICH STUDENTS CAN INVENT INSIGHTFUL PROOFS IN GEOMETRY

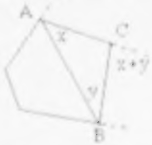
Nurit Hadas and Abraham Arcavi - Weizmann Institute of Science - Israel

The following is one of many activities we have developed to learn geometry in a computerized environment. It is being tried in several classrooms and with individual students. After students know that the sum of a triangle is 180°, they explore the sum of the internal angles of a quadrilateral, a pentagon, a hexagon, etc. After they have spent some time drawing and measuring, they usually have no problem in inducing that adding a side adds 180° to its sum. In order to probe what students regard as a convincing explanation, we ask "why?". The following is an example of an interesting response.



> In order to add a new side to a polygon, I "break" a side into two new sides (as in the drawing) The 'amount' of angles added is the 'amount' inside a triangle

Then students are asked to check the sum of the external angles of a quadrilateral, and to predict whether this sum increases or decreases with an increase in the number of sides of the polygon. Most students predict that the sum increases (because the number of angles increase). Some say that the sum decreases because each external angle becomes smaller. Usually, nobody predicts that the sum does not change. We ask students to collect data (measurements) - and they are very surprised by the result. It is this surprise that invites the need and search for an explanation. The following is what we heard from the same student, who built her new argument on her previous one.



> When I made the quadrilateral into a pentagon, the external angle at A decreased by, say, x The same happened with the external angle at B, which decreases by, say, y However, a new external angle appears at C, which is exactly x+y

Thus she seemed to understand why the sum remains unchanged when we increase the number of sides. The value of these examples might seem to be no more than an 'existence proof' that some students are capable of generating insightful explanations. However, we think there is more to it. What are the characteristics of the environment which may support students to generate proofs? First, the activity is posed as an exploratory task, rather than "prove that...". Second, students have a handy tool to draw and to measure and thus have the means to generalize, to conjecture, to contrast intuitive predictions with actual results, and to develop a feel for the need to justify. Third, students are nudged to explain the underlying "mechanisms" and thus learn that their need is not only respected, but also it becomes an integral part of what learning geometry might be. We propose that most of geometrical content can be learned if we explicit these characteristics, as opposed to the conventional chain of stated theorems followed by their formal proofs.

# ON THE KNOWLEDGE OF HIGH SCHOOL MATHEMATICS TEACHERS FOR TEACHING PROBABILITY

**Avigail Hartman**

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Probabilistic terms and ideas often appear in the media and in a wide range of diverse disciplines. This situation has generated a growing tendency to teach probability in high schools, and concurrently stimulated numerous studies on various aspects of probability comprehension (Shaughnessy, 1992).

Studies conducted on children and prospective teachers show that the development of probability understanding and probabilistic intuition is an integral part of the mental development of a person. People are often confronted with the need to make decisions under uncertain situations, and in resolving them, develop 'probabilistic' strategies, conceptions and misconceptions. The formal study of probability in school, enhances the ability to comprehend and apply probabilistic models.

The teacher of probability, who plays an important role in this learning process has not been studied sufficiently. This research report focuses on teachers' knowledge of their students actual problem solving strategies. The teachers were asked to solve various problems in probability. In addition, they were requested to solve the same problems in ways they presume their students would.

This methodological approach serves a dual purpose: measuring the teacher's subject matter knowledge and learning about the teachers' pedagogical content knowledge (Even & Tirosh, 1995).

The results of the study provide evidence that teachers themselves encounter difficulties in probability problem solving. They tend to use different problem solving strategies themselves than what they attribute to be the expected problem solving strategy of their students. It seems that they assess their students problem solving strategies as being more procedural and less conceptual then their own.

## References

Shaugnessy, J.M. (1992). Research in probability and statistics: Reflections and directions. In D.A. Grouws (Ed.) Handbook of Research on Mathematics Teaching and Learning, 465-494, New York: Macmillan.

Even, R. & Tirosh, D. (1995). Subject matter knowledge and knowledge about students as sources of teacher presentations of the subject matter.

# FACTORS INFLUENCING THE LEARNING OF MATHEMATICS AMONG VOCATIONAL AND TECHNICAL SCHOOL TEACHERS

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The research describes and gives a picture on adults life and experiences as students in the university. Majority of the students are from Vocational and Technical schools enrolled under the Bachelor's Degree in Technology Education. As a requirement, these students had to take mathematics courses, seen as the mostly feared, abstract, much-talked about subject in campus. The study investigates students perceptions, attitudes and learning skills towards the learning of mathematics. The aim of the study was based on the following reasons:

i) to identify problems faced in the learning of mathematics
ii)do adult students have the necessary mathematics foundation required by the university
iii)do adult students have different characteristics in the learning of mathematics
iv)what topics are relevant or irrelevant in their respective discipline and courses.

A descriptive research was conducted comprising of questionaires based on motivational factors, learning content and instruction as well as personal factors to determine the adults learnings styles, attitudes and aspirations. The Likert Scale was used and the data was analised using SPSS. Results from the questionaires showed that adults students showed high motivation and strong sense of determination to succeed. They favoured strongly to courses that had practical activities that would give them confidence and better experience for their respective vocational and technical work profession. This agrees with Cole (1980) suggesting that course content and instructional techniques enhance student motivation and achievement. It is important that students and instructor develop good relationship among one another, are receptive towards their comments and needs and are willing to help them. Effective teaching is closely linked with the compatibility between students and the method of instructors in the university (Wankowski, 1970).

## References

Apps, J. W. (1977) *Study skills for adults returning to school*. New York: McGraw-Hill

Cole, H.P. (1980) Principles and Techniques for Enhancing Motivation and Achievement of Engineering Students. *Proceedings 1980 College Industry Education Conference*

Knowles, M.S. and Associates. (1984) *Andragogy in action*. San Francisco: Jossey-Bass Publishers

Wanskowski J. A.(1970) *A Random Samples Analysis: Motives and Goals in University Studies*. Birmingham: Educational survey, University of Birmingham

Wieschuegel,R.E.(1978) *Measurement of Cognitive Achievement in Continuing Education in Engineering*. Unpublished Ph.D dissertation , George Peabody College for Teachers, Nashville, TN

Wlodkowski, R. J. (1985) *Enhancing adult motivation to learn*. San Francisco: Jossey Bass Publisher

# How Students Use the Properties Found in Geometry Software to Write Proof

## - Analysis of Students' Activities by using Cabri -Geometry -

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The use of software helps students to conjecture the properties of geometric figures and theorems (Chazan, 1989,etc.). Usually in a secondary school geometry, conditions and a conclusion are given and students are asked to draw logically the conclusion based on these conditions. In these situation, students have seemed to have no motivation to make proof and to have difficulties to write proof. In this research, an effect of property finding activities by software on writing proof was examined.

Method : Subjects are forty-three students in 3rd grade of a junior high school (14,15 years old). There are two sessions. In the first session, students are asked to explore and write properties of geometric figure described in a problem by Cabri-Geometry. Each pair of students use one computer. In the second session, students are asked to write proof, individually. Their worksheets used in both session are analyzed focusing what kinds of geometrical properties they find and how they use them in their proof.

Table Relation between found and used properties and proof

| | correct proof | incomplete proof | t-value |
|---|---|---|---|
| no. | 15 | 22 | |
| No. of useful properties for proof found(b) | 1.73(± 1.39) | 1.04(± 1.40) | t=1.46 |
| No. of used properties of (b) in proof | 1.73(± 1.39) | 0.77(± 0.99) | t=2.53 * |

* : $p<0.05$

Results (1) Properties found were grouped into (a) a conclusion which is usually given in a proof problem, (b) useful properties as a part of logical steps in proof and (c) useless ones. Twenty-seven (62%) students found a property as a conclusion. (2) Fifteen (34.9%) students wrote correct proof and twenty-two (51.2%) students wrote incomplete one. Six(13.9%) students were not able to write proof. The table indicates that *Students who wrote correct proof used more properties (1.73) as a part of logical steps in proof than those (0.77) used by students who wrote incomplete one.* This difference is statistically significant ($p<0.05$). (3) In the first session, Hiromi wrote a conclusion and the reason why it was true by himself.

Conclusion A positive effect on writing proof by properties finding activity by geometry software was identified in our experimental teaching trial. Detail observation will provide a close picture on how students use properties found to write proof.

References Chazan ,D and Houde,H(1989), How to Use Conjecturing and Microcomputers to Teach High school Geometry, NCTM Reston, VA.

Title **A metaphor for teaching negative numbers: The Parking Garage Paradigm**

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This article reports one component of an on-going longitudinal research project. The purpose was to create a context for introducing addition of integers that would be meaningful to children. An open-ended problem task, the Park Garage Activity, was developed by the researcher, based on ideas from Robert B. Davis about the necessity for the learner to develop "assimilation paradigms", powerful metaphors for thinking, as essential steps in building knowledge of mathematical ideas. The study took place in a private school in Rio de Janeiro, Brazil. Four students, each about 10 years of age, were videotaped during a forty-five minute problem-solving session. The activity involved controlling the quantity of cars that entered and exited from the parking garage over specific time period. There were four roles to be played in the Parking Garage Activity: (1) the guard - to keep order, indicating when cars may enter or exit the garage; (2) the entrance controller - to control the number of cars that enter into the garage; (3) the exit controller - to control the number of cars exiting the garage, and (4) the overseer - who verifies the total number of cars in the garage at any given time. Each student was assigned one of the four roles and expected to record the relevant data from each action of the problem on a blank sheet of paper. During the activity, each student was given the opportunity to play all of the roles. The students initially came up with a variety of different notations for describing the actions of the cars. Two of the children, in their initial observations, spontaneously used the idea negative and positive numbers in the resulting for the garage. After finishing the Parking Garage Activity, the students were given a chart and asked to determine an appropriate way to organize and describe the data, including the number of cars entering and exiting, along with the resulting state. As the students compared their individual solutions, discussing the various notations that they had used, they agreed that the most effective method of notation was signed numbers, with positive and negative denoting the relative state of the garage.

Conclusion: Different cultures require different activities to enable children to development such an assimilation paradigm for the same mathematical idea. In the United States, for instance, one might use temperatures above and below zero. In rural Africa, there was developed a game which used "Pebbles in the Bag". In the urban city of Rio de Janeiro, Brazil, we have a parking garage. Our research has focused on the same ideas, using different metaphors that respect the differences among cultures.

References

- Brazil, Circe N. Vital (1976) - Proposta Curricular de 6ª Série - Publicação do Laboratorio de Curriculos, Secretaria Estadual de Educação
- Davis, Robert B. (1984) Learning Mathematics: The Constructivist Aproach to Mathematics Education. Norwood, NJ: Ablex Publishing Corp.
- Maher, Carolyn, Martino, A. & Alston, A. (1993, July). Children's Construction of Mathematical Ideas. Paper Presented at the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA), Brisbane, Australia

# WHAT DO YOUNG CHILDREN UNDERSTAND ABOUT DIVISION?

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Much research on children's numerical understanding has focused on their computation ability. However, children can understand some logical principles of arithmetic operations even before being taught the corresponding sums. Correa (1995) showed that 6 year olds can understand the inverse relation between the divisor and the quotient with discontinuous quantities in non-computational sharing problems. We examined children's understanding of this relations with discontinuous and continuous quantities. This extension places the problem in the domain of fractional numbers.

Children (age range 5.5 to 7.5) who had not been taught division at school were presented with partitive and quotitive division situations. In the partitive situation, they judged the relative number of a quantity of fish or the size of the piece of fishcake that each cat would receive when the number of cats varied and the dividend was constant. In the quotitive situation, they judged the relative number of recipients when the shared quotas varied.

One-third of the 5.5 year-olds and almost all of the 7.5 year-olds understood the inverse relations in partitive division problems; no performance differences were observed between discontinuous and continuous quantities. Quotitive problems were significantly more difficult, but only when presented with discontinuous quantities. Just over a half of the children solved the discontinuous quantities quotitive tasks at the age of 7.5 but nearly two-thirds succeeded in the continuous quantities problems. Children who performed beyond chance level were able to explain that, the more the sharers, the smaller the quota and vice versa.

We conclude that young children understand some of the logical relations in division and suggest that schools should consider how this knowledge can be capitalized.

Correa, J. (1995). *Young Children's Understanding of the Division Concept*. University of Oxford: Unpublished PhD thesis.

# CHANGES IN TEACHERS' BELIEFS OF MATHEMATICS TEACHING AND LEARNING

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A growing knowledge of the role that teachers' beliefs play in teaching has led to the question of how teachers' beliefs of mathematics and mathematics teaching and learning can be influenced and enriched. How do teachers' beliefs evolve? Quite a lot of research has also been made about changes in teachers' beliefs. Research literature has showed that the development of beliefs over time would appear to be difficult to predict, control or influence. This is because the development of a given teacher's beliefs of mathematics teaching and learning is influenced by the personal experiantial background of that teacher, including his/her professional and educational experiences, and how these are interpreted and internalized by the teacher. This means that change in teacher must occur from within and cannot be imposed from above: not trying to change beliefs in order to have the 'right' effect but rather as a means to throw light on beliefs, beliefs-on-practice and the innovation we are striving for. In this way it would be possible to help teachers and prospective teachers to become reflexive and self-conscious of their beliefs and present 'objective' data on the adequacy of these beliefs.

In this presentation the author will discuss changes in the beliefs of the Finnish comprehensive school mathematics teachers in the first half of the 1990s. The empirical data has been collected in two phases: the first study took place in 1990 and the second one five years later in 1995. Altogether 68 mathematics teachers from the grade 9 participated in both studies and 15 of those teachers were same teachers. The main data was collected by using the teacher questionnaire which consisted of the belief inventory and a lot of background information. Furthermore, some interview material was collected in the spring 1996. The results suggest that mathematics teachers hold multidimensional beliefs as to the nature of mathematics teaching and learning. However, teachers' beliefs seem to change in some degree as the result of the teachers' natural professional development but simultaneously they have some strong core beliefs in which they absolutely persist.

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# THE USE OF SPREADSHEET IN PROFESSIONAL CONTEXTS: EPISTEMOLOGICAL OBSTACLES AND AMPLIFIED COMPETENCIES

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Many studies have focused on the use of eletronic spreadsheets as a versatile device which offers conditions to identify, manipulate, explore and represent quantitative relationships between entities (Kieran, 1987; Rojano & Sutherland, 1991). Spreadsheet resources allow the production of strategies and meanings in a wide range of mathematical contents. The use of spreadsheets to make explicit the procedures involved in problem-solving may lead to exploration of new concepts and integration of knowledge.

This study consisted on an investigation of the cognitive aspects involved in the use of spreadsheets as a technological tool in a professional context. We examined the socio-professional context where the spreadsheets are often used, what types of professionals use them and how they are used. The subjects solved similar arithmetical and algebra word problems, from the point of view of their mathematical structure. The contents of both types of problems were either familiar or unfamiliar in relation to the context of the professional environment: economists and accountants. Thirty employees participated in the study. In addition to spreadsheets, the subjects were given hand calculators and paper and pencil.

The analyses showed that: (1) Familiar and unfamiliar contents consisted in the problems had no significant effects on the subject's performance; (2) Spreadsheets users had two different levels of proficiency: some used the environment just to manipulate numerical data, while others used it as a tool to explore concepts and integrate knowledge for problem-solving; (3) The performance of the subjects restricted to manipulate data was not significantly different from non-users, in relation to the problems presented; (4) Epistemological obstacles were identified in the group who use the spreadsheet just to enter data in worksheets without explore its concepts.

References

KIERAN, C., GARANÇON, M. et BOILEAU, A. (1990) Introducing algebra: A functional approach in a computer environment. *Proceedings of the XIVth International Conference of the Psychology of Mathematics Education* (vol II, pp. 51-58). México.

ROJANO, T. & SUTHERLAND, R. (1991) Symbolising and solving algebra word problems: the potencial of a spreadsheet environment. *Proceedings of the XVth PME*, (pp. 207-213). Assisi.

# STUDENTS DEPENDENCE ON SYMBOLIC VARIABLES IN FUNCTIONS

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There are many standard procedures in Calculus in which students perform well, however research has shown that most students do not apprehend the character of functions and their crucial rôle in Calculus. What the standard procedures mainly depend on is the presentation of the function to be in terms of an algebraic expression involving one symbolic variable, $x$ say. Well - known techniques allow students to explore properties of the function. Hence this type of representation is often dominant in exercises about functions. This leads to a restricted experience in working with functions with the following consequences: (a) the variable $x$ may be regarded as an unspecified number or as "a space" where any particular chosen number may be placed [Arcavi and Schoenfeld]. In this case, the idea of $x$ being a "general" number is not reached, and the "objectification" of the function is unlikely to be satisfactorily realised. (b) Often the manipulation of $x$ in solving a question is difficult or inappropriate. (e.g. a question might involve a general function). The student lacks alternative strategies to obtain information because he / she is not accustomed to or does not think it is justified to work with other representations of functions, especially graphical. This results in a lack of "sense of functions", a phrase borrowed from [Eisenberg]. (c) Students tend only to work with functions descriptively, that is they are given functions for which they have to examine properties. This does nothing to develop skills of identifying, forming and using functions.

Is it feasible, then, to design and use exercises which avoids explicit representations of functions as algebraic expressions, and if so how effective will these be in broadening the students' comprehension of the significance and character of functions? This research line seems relatively untouched.

In my presentation, I will describe a pilot test given to a class of first year university students (studying Economics). All the problems given involved functions extracted from physical contexts, but none require forming explicit algebraic expressions. The type of the problems are very different, varying in how geometric the context is and the level of abstraction needed; the latter depending much on how descriptive or applicative each problem is. About half the students were able to describe the main properties of a function corresponding to a given relationship in a geometric framework without using an algebraic expression. Very few did well for the problems which required comparisons, identification or application of functions. These results suggest that it is feasible and useful to present students with examples of examining given functions in a non - algebraic way, especially from geometric contexts. A question remains; can we eventually direct this new kind of experience to other activities, such as problem solving, requiring more creative use of functions?

A. Arcavi & A. Schoenfeld (1987): "On the meaning of variable", *Mathematics Teacher*, 81(6).

T. Eisenberg (1992) : "On the development of a sense for functions", *The Concept of Function - Aspects of Epistemology and Pedagogy*, Ed. Dubinsky, G. Harel (Eds.), MAA Notes 25.

# PRECURSORS OF ADDITIVE COMPOSITION OF NUMBER

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Understanding additive composition of number (ACN) is implicated in the ability to combine coins of different denominations. There are two views of what are the important precursor skills to ACN.[1] One, is that ACN develops from experience of addition. The development of the counting-on strategy for solving addition problems is seen as the crucial step (Nunes and Bryant, 1996).

An alternative account of the development of ACN stresses the development of continuation of counting, an early counting skill that enables children to count up from an arbitrary number in the list. While both counting-on and ACN develop from continuation of counting, there is no necessary link between them.

A cohort of 168 children between 4 and 7 years of age were tested three times during the school year on tasks assessing their ability to continue counting, their strategies for solving addition problems, and their understanding of ACN.

No child, at any point, succeeded on ACN without also being able to continue counting. In contrast, despite being more likely to pass ACN if they used counting-on, there were substantial numbers of children who passed ACN but did not show any evidence of counting-on.

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[1] Nunes, T. and Bryant, P. (1996). *Children Doing Mathematics*. Oxford: Blackwell.

# FROM LEARNING GEOMETRY TO LEARNING TO TEACH IT: AN ANALYSIS OF A CASE-STUDY WITH PRIMARY STUDENT-TEACHERS[*]

Ana LOBO DE MESQUITA, Louis ROYE
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Students enrolled in pre-service primary education, at least in France, have in general humanist backgrounds, where mathematics take a small part. But, to teach mathematics at school, a rich and dynamic knowledge concerning basic mathematics is clearly required. How to conciliate these opposite facts? How to enable student-teachers to structure mathematical knowledge in conceptual fields (Vergnaud, 1991), a way to understand complex points of their future teaching?

We hypothesize that such a structuration becomes possible by the utilization of some methodological principles: a first principle is based on the problematisation of situations, in the sense of a transformation of questions in problems to solve, which presupposes that the student-teacher has the opportunity to be confronted with such situations; a second one lies on the analysis of: i) mathematical tasks, ii) pupils's errors.

To test this hypothesis, we develop an intervention, choosing a small but central point in the mathematical program, the plane representation of spatial objects. Two groups of student-teachers, volunteers, participate, during eight weeks of their education in IUFM, in 1994/95 and 1995/96 in: a) "problematisation worshops" concerning representation of geometrical solides, where manipulation and use of different forms of representation takes an important part; b) "observation-and-analysis sessions", centered on the analysis of video sequences, where student-teachers were asked to analyse tasks and pupils'productions.

The analysis of all the work done by student-teachers suggests important progress: a deep analysis on the strategies choosen by teachers, a real understanding of some errors made by pupils. The use of manipulative materials is considered as something essential to the understanding of the relationships between space and plane, enabling student-teachers to anticipate pupils' difficulties.

**References**

MESQUITA A.L. et al. (1997) *Apprendre la géométrie pour apprendre à l'enseigner : analyse d'une expérience avec les futurs professeurs des écoles*. Research Report, Projet # 3/94/004, IUFM NPdC, Lille

VERGNAUD G. (1991) La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques*, *10* 2/3, 133-170.

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# STUDENTS LOOKING FOR SIGNS — A STRUGGLE FOR SENSE MAKING IN MATHEMATICS

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What is an equation? What is a quadratic equation? What is a system of linear equations? What do they look like? What are they for? How do we deal with them?

We understand the world by classifying the objects we interact with, and thus creating categories. According to an objectivist view of knowledge either we know *a priori* all the elements of a certain category or we decide, on the basis of the objects' inherent properties, if they belong or not to a certain category. The concept formation would then depend on the knowledge of those properties which had to be previous. The concept would indeed require the concept definition.

Thinking otherwise I want to argue that this is not the way we manage to organise our experiences. The formation of concepts surely involves the identification of common properties between certain objects; but it also derives from the recognition of the differences between those particular objects and furthermore between those and other objects. Concepts are formed against other concepts.

Moreover the properties that we ascribe to an object are not always inherent to the objects but instead a product of our ways of relating to them. Rather than being rigidly defined, concepts emerge from our experiences, through our interactions with the environment and other people.

A mathematics teaching approach based on the sharply defined sequence — contents presentation, exercises and practice, evaluation — meets the objectivist perspective on the formation of concepts. When exposed to this kind of teaching, students, even if they are successful in the short term, do not really understand the concept and when confronted with new situations seem to be impotent to make sense of them. Their temporary success is built on their accommodation to 1) the fact that mathematics contents are treated as if they belong to some well defined compartments, to which certain limited slots of time are dedicated; 2) the routine of assessing a certain number of items as a follow-up of the class work on a certain topic; 3) the picking up of signs that guide them in the identification and recognition of the techniques to be used within a limited universe of mathematical objects.

In this presentation we'll *hear* a group of college students in a calculus course class while they work on a problematic situation. I want to draw attention to the way they desperately look for the kind of signs mentioned above. Undoubtedly those signs are a reflection of a twelve years past in school mathematics, but they are also the signs without which students seem to be unable to find their way within their mathematical worlds.

References
Lakoff, G. & Johnson, M. (1980). *Metaphors We Live By*. Chicago: University of Chicago Press.
Saussure, F. (1966). *Cours de Linguistique Générale*. New York: McGraw Hill.
Skemp, R. R. (1987). *The Psychology of Learning Mathematics*. London: Lawrence Erlbaum.

# PSYCHOGENETIC DEVELOPMENT AND PROBLEM SOLVING IN MENTAL HANDICAP

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According to Inhelder and collaborators, when solving a problem the «psychological subject» generates a mental representation of it, which guides the construction of the procedure leading to a response. Although structures and procedures are considered as being two related components of all cognitive activities, some controversy exists about the influence of the psychogenetic development on the construction of these procedures.

Considering this, we proposed to study how mentally handicapped subjects solve problems, focusing on 3 questions: a) the relationship between psychogenetic development and type of strategy; b) the relationship between mental representation, procedure and response; c) a comparison between the strategies used mentally handicapped subjects and normal children. According to our main hypothesis, the strategies used by this population are different from those used by normal subjects of the same age, but similar to the ones used by younger children with the same psychogenetic development.

Our study involved 30 subjects (9 in the 2nd grade, 11 in the 3rd grade and 10 in the 4th grade) from a Special Education School. After they were evaluated with Piagetian tasks, they were asked to solve, in a random order, six word problems involving multiplication. Concrete material was available. They were also asked to explain how they interpreted and solved them.

The results showed that they used several strategies (chance, counting, addition, subtraction, multiplication, division), which are similar to those used by normal children, "counting" being the most frequently used and "division" the least used. In 97% of the cases the correct representation is associated with the correct answer. Data also point out to a close relationship between cognitive development and generation of procedures, once subjects who showed a higher level of development in the operatory diagnosis (*diagnostic opératoire*) used more complex and diverse strategies.

## References

INHELDER, B.; CELLERIER, G. (1992). *Le cheminement des découvertes de l'enfant.* Paris, Delachaux et Niestlé.

MORGADO, L. (1991). Word problems – the construction of multiplicative structures. In F. Furinghetti (Ed.), Proceedings of the 15th PME Conference (vol III), 25-32.

PIAGET, J.; INHELDER, B. (1959). *La genèse des structures logiques élémentaires.* Paris, Delachaux et Niestlé.

PIAGET, J.; SZEMINSKA, A. (1941). *La genèse du nombre chez l'enfant.* Paris, Delachaux et Niestlé.

VERGNAUD, G. (1983). Multiplicative structures. In R. Lesh e M. Landau (Eds.), *Acquisition of mathematics concepts and process.* New York, academic Press.

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# THE USE OF STRING FIGURE GAMES IN JUNIOR SECONDARY MATHEMATICS CLASSROOMS

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## AIM AND CONTEXT OF STUDY

Games have been used in the teaching and learning of mathematics over the years, and this use has increased in the recent past (Vithal, 1992: 178-179). Kirkby (1992: 5-7) has documented the variety of uses of games. A few of the uses are:

-learn the language and vocabulary of mathematics,
-develop mathematical skills,
-devise problem solving strategies,
-be the generator of mathematical activity at a variety of different levels.

Many of the games that are mentioned in the literature that refers to the use of games mention games that are general i.e. not specific to a particular cultural group or games that are traditional in nature.

This paper reports on some of the results of a study which is currently going on in a number of South African Secondary Schools on string figure games. This study looks at whether games can find use in mathematics classrooms, especially games that are cultural and traditional in nature. Some of the specific questions of the study are:

1. What are the possible uses of string figure games in mathematics classrooms?
2. Can the students understand the rules of string figure games and be able to apply them in mathematical situations in a relatively short period of time of a mathematics lesson?
3. Does prior familiarity with the game makes it easier for the identification of mathematical concepts that are related to the game?

## METHODOLOGY

Lessons (interactive) based on string figure games were presented to three secondary schools. The schools were not far from each other (about 5 kilometres apart), making it possible in terms of comparisons with respect to prior experience and cultural setting. Students were given pieces of string and allowed to try any string figure game that they could make. After a period of about 10 minutes different gates arising from string figure games were demonstrated to the students. This was followed by a questionnaire on the use of string figure games.

## FINDINGS

The results of the study indicate that most of the students had not played games in the mathematics classroom, in those cases where they had none of the games were traditional. More students in every group found it difficult to understand the rules of the game, although further enquiry revealed that most students equated understanding the rules of the string figure games to making many gates. Despite the fact that most students had not played a game in class before, many indicated that games would be very helpful in learning mathematics.

## REFERENCES

Kirkby, D. (1992). *Games in the Teaching of Mathematics*. Cambridge: Cambridge University Press.

# THE COMPREHENSION OF GRAPHS AND SCHOOL MATHEMATICS[1]

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Nowadays, most of the information in the newspapers and magazines is given by means of different kinds of graphs. A well-informed citizen, who usually follows the political and economical scenery, must be able to interpret and handle all these data.

But at what extend does the math's experienced at school provide the basis for the comprehension of such graphs? In general, Brazilian students do not have any contact with graphic representations until the last years of middle school (13-15 years), and, therefore, they may present difficulties in the complete understanding of the information given through graphs.

The purposes of this research are to analyze some graphs presented in usual communication vehicles, and investigate how Brazilian high school students interpret them.

The sample for this study was composed of three classes of 15 to 17 year old students, from a state school in Rio de Janeiro, a total of 88 students.

In the first step, we have selected some interesting graphs from the most popular newspapers and magazines in Rio de Janeiro, and analyzed if their layout could really help the comprehension of the information intended to be given.

Next, some questions have been elaborated about each graph, in order to investigate if the students in the sample were able to understand the information contained in the graphs, and, if possible, how this was influenced by the math's they learned at school. We were also interested in observe if the visual representation could lead some students to misunderstand the information.

The graphs chosen for the investigation involved percentages, negative numbers, comparison of increasing slopes, and the comparison of two different graphs representing the same data.

The first results show that:

- students do not pay attention if the graph presents errors: they just believe in what is stated;
- in general, students do not use concepts learned at school when interpreting a graph;
- some graphs are difficult to be understood, and therefore, do not help at all the acquisition of the information;
- when the graphic representation is clear and straightforward, the students can easily get the desired information.

So, teachers should be aware that students must be prepared for the comprehension of graphs, through the experience of activities exploring the visual and the linguistic aspects involved in such representations.

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[1] This research was sponsored by CNPq (Brazil)

# MATHEMATICS - IMAGES AND CULTURE OF LEARNING

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The present study is oriented by questioning the role of the past mathematics experience in students' learning and their relationship with mathematical tasks in classroom - how images about learning mathematics and the subject matter influence students atitudes towards mathematics and their learning process? How do students change their images about Mathematics?

In this research that is running since 1995/96, our focus is the experience with math that was recorded and crystallized in the memory of students and is presented as images. Experience here is a started point to understand how knowledge is formed. We looked for the meaning that students give to their experiences with math following the conceptual framework of images considered by Elbaz (1983; p. 254) as "a brief descriptive and sometimes metaphoric statement". And also, how this images have expression in action during classroom interactions with mathematics.

The images allows us, to study: a) the socio-affective aspects that are related with learning; b) the practical knowledge that students have c) how they project their images in action.

Interviews, class observations, oral narratives, memos, projective task and mathematic material produced by students provided data for analyses, in this study. Students of three classes, involved in a project that aim to create a curricular alternative, were the participants.They gave meaning to their past experiences and present actions with mathematics in the context of school and out side of it. The study gives some evidence how different images and beliefs about mathematics and its learning, that these students bring to the classroom, affects their relationship with mathematics in classroom situations and their learning process. The implications of students images in the curriculum building, and learning culture, will be debated.

## References

Abreu, G. C. P. (1993) *The relationship between home and school Mathematics in a farming community in rural Brazil.* A dissertation for the degree of Doctor of Philosophy at the University of Cambridge.

Clandinin, D. J. (1986). *Classroom Practice: Teacher images in action.* Philadelphia: The Falmer Press.

Elbaz, F. (1983). *Teacher thinking: A study of Practical Knowledge.* New York: NIchols Publishing Company.

# OPERATIONAL, PROCESS AND STRATEGIC ABILITIES IN THE LEARNING OF ALGEBRAIC LANGUAGE - A CASE STUDY

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In this work we study from a cognitive perspective the operations, processes and strategies used by the subjet when learning, that is when acquiring, organizing, formulating and recovering knowledge of algebraic language, while at the same time increasing control and awareness of the cognitive processes when learning algebraic language.

Our work connects these two fields, the cognitive one and the metacognitive one, focusing on 12-13 years old students' habitual school environment, exploring the approach to algebraic language, while emphasizing the possibilities that arise when new teaching resources and means of observation are introduced.

We have proposed the use of the so-called Visual-Geometric Representation System by which any expression codified in this system may be used for both syntactic and semantic work, based on operations with magnitudes constitutes a culturally, didactically and mathematically meaningful representation system.

We emphasize the results of the case study where a student's difficulties are evaluated when using two juxtaposed systems of representation in order to learn algebraic language. Analysis is made of data compiled from a questionnaire (pre-test), classroom workbooks, questionnaire (post-test), revision activities undertaken by the student outside the school environment, and videotaped sessions (case studies).

In the results we can note integration of aspects concerning mathematical contents and aspects concerning the relations between the student and the acquisition of algebraic language, by juxtaposing the representation systems, both of them confirm, on one part, beliefs about the low level of algebraic language students possess at the stage in their schooling and the positive contribution played by the juxtaposition of the representation systems in the acquisition of algebraic language, promoting abilities in operations, processes and strategies.

## References

Davis, R. (1984) *Learning Mathematics. The Cognitive Science Approach to Mathematics Education.* Australia Croom Helm.

Kieran, C. (1992). The learning and teaching of school algebra. En Grows, D.A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning.* Macmillan Publishing Company. New York.

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# HOW DO YOU KNOW WHEN YOU'RE DONE? STUDENTS REFLECT UPON THEIR PROOF MAKING

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"...the major emphasis is placed on a body of theorems to be learned rather than on the *method* by which these theorems are established." - 13th yearbook - NCTM 1938

Studies report that a majority of students are either unable to construct formal proofs, or hold alternative notions of what constitutes a valid proof. Studies have suggested that students investigate meaningful situations where proof can arise as a response to the situation. Is it possible to "lead" students to appreciate a need for formal proof? Will such experiences lead students' to use formal proof? And if so, how do students move beyond algorthmic conceptions of formal proof?

This paper will speak to these points within the context of an ongoing longitudinal study into children's mathematical thinking. For the past eight years, students in this study have interacted in classroom contexts where they were encouraged to build justifications for their conjectures. Their mathematical thinking has been observed, videotaped, and analyzed in fine detail throughout this time. The students' most recent work suggests that the forms their justifications take are embedded in the norms of their mathematical community.

Yet these norms were an outgrowth of the students' prior activity with proof. This duality will be examined through analysis of videotapes of past and present work. This paper extends previous research into what must accompany "meaningful situations," and supports the idea that a student's notion of proof is profoundly influenced by a reflexive process of goal formation. This paper presents evidence that proof is located for students within a framework of constructive activity. Implications for reform initiatives which under-theorize students' conceptions of their own activity will be explored.

Battista, M, and Clements, D. (1995). Geometry and Proof. Mathematics Teacher, Vol. 88, January 1995, 48-54. Reston, Virginia: NCTM.

Ernest, P. (1995). The Mythic Quest of the Hero: A Semiotic Analysis of Mathematical Proof. Paper presented at the 19th International Conference for the Psychology of Mathematics Education. Recife, Brazil.

Fawcett, H.P. (1938). The Nature of Proof. Thirteenth Yearbook of the National Council of Teachers of Mathematics. New York: Columbia University Bureau of Publications.

Leont'ev, A.N. (1981). The Problem of Activity in Psychology. In Wertsch, (Ed.) The Concept of Activity in Soviet Psychology. Armonk, NY: M.E. Sharpe.

Maher, C. A., & Martino, A. M. (1996). Young Children Inventing Methods of Proof: The Gang of Four. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), Theories of Mathematical Learning. Hillsdale, NJ: Lawrence Erlbaum Publishers.

Maher, C.A., Martino, A.M. & Pantozzi, R.S. (1995). Listening Better and Questioning Better: A Case Study. In L. Meira and D. Carraher (Eds.) Proceedings of the 19th International Conference for the Psychology of Mathematics Education. (Vol. 3) 82-89. Recife, Brazil.

Yackel, E. and Cobb P. (1995). Classroom Sociomathematical Norms and Intellectual Autonomy. In L. Meira and D. Carraher (Eds.) Proceedings of the 19th International Conference for the Psychology of Mathematics Education. (Vol. 2) 264-271. Recife, Brazil.

# UPGRADING THE PRACTICAL PEDAGOGICAL KNOWLEDGE OF MATHEMATICS TEACHERS THROUGH MEDIATED-LEARNING

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**ISRAEL**

This paper presents a synthesis of recent work, by two researchers. A major emphasis in our work is on research and development program for mathematics teachers in elementary schools. The program for upgrading the practical pedagogic knowledge of mathematics teachers is based on four areas:

1. The theories of mediated learning and cognitive modifiability (Feurstein, Rand, Hoffman & Miller, 1980).

2. The development of pedagogical personal practical knowledge as a basis for teaching, e.g., knowledge of subject-matter, pedagogics, didactic and curriculum knowledge, and knowledge about the "self" (Shulman, 1987; Connelly & Clandinin, 1991).

3. "Measure and Know" - Mathematics program for young children (Nir-Gal, Millet and Matalon, 1996) which includes research models and measurement problems of length, volume and weight.

4. Ten years of experience in practicum in a unique model of disciplinary didactic (mathematics) workshop at the Achva College, Israel.

The program has been developed as a part of the national project "Tomorrow 98" according to Van-Hiele's theory (Patkin 1990). We studied 30 mathematics teachers and followed the whole process. We used open-ended questionaires in three different stages (pre, post and retention). The questions were about geometrics concepts, research concepts and mathematics assessments. The research methodology was naturalistic, qualitative and interprentative and was based on the case study method.

The results indicate a change in the practical knowledge of math teachers. There was an improvement in their knowledge of geometry and research concepts. They succeeded in creating alternative assessments in geometry. Weisglass (1994) mentioned that changing mathematics teaching means changing ourselves.

We assessed the contribution of the program to the professional development of the math teachers.

## References

1. Connelly, F.M. and Clandinin, D.J. (1991). Teachers as Curriculum Planners Narrative of Experience. Baltimore, University Park Press.
2. Feurstein, R., Rand, Y., Hoffman, M.B. and Miller, R. (1980). Instrumental Enrichments for Cognitive Modifiability. Baltimore, University Park Press.
3. Nir-Gal, O., Millet, S. and Matalon, Y. (1996). Measure and Know - Mathematics Program. Ministery of Education and Culture, Israel.
4. Patkin, D. (1990). The Utilization of Computers: Its Influence on Individualized Learning Pairs versus Individualistic Learning on the Perception and Comprehension of Concepts in Euclidean Geometry at Various Cognitive Levels within High School Students. Ph.D. thesis, Tel-Aviv University, Israel.
5. Shulman, J.H. (1987). Knowledge and Teaching Foundation of the New Reform. Harvard Educational Reviews **57**(1): 1-21.
6. Weisglass, J. (1994). Changing Mathematics Teaching Means Changing Ourselves. Implications for Professional Development for Teachers of Mathematics. 1994 Yearbook, Reston NCTM, pp. 67-78.

# YOUNG CHILDREN'S INTUITIVE STRATEGIES FOR MULTIPLICATION AND DIVISION WORD PROBLEMS IN A PROBLEM-CENTERED APPROACH

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This report is based on findings of a study that examined the intuitive strategies used by young children when solving multiplication and division word problems. Essentially, this study was a replication of related research carried out in other parts of South Africa as well as internationally (Murray, Olivier and Human, 1989; 1991; 1992; 1993; Cobb, Wood, Yackel, Nichols Wheatley and Perlwitz, 1991; Fennema, Carpenter and Petersen, 1991; Mulligan, 1992). It involved a detailed qualitative analysis of the different intuitive strategies used by the children when solving multiplication and division word problems, in order to determine the relationship between semantical structure and strategies used; as well as the intuitive models used by the children.

Nineteen pupils from the Junior Primary Phase participated in a 10 week study during 1994, in which, the instruction was generally compatible with the principles of Socio-Constructivism and the Problem-Centered Mathematics Approach. The word problems which were classified according to semantic structure consisted of five multiplication and five division problem structures. The results indicated that 76% of the sample were able to solve the ten problem structures intuitively by the end of the experimental period. The children, who had not received any form of instruction on multiplication and division algorithms or associated concepts, used a range of informal strategies which were obviously based on their knowledge of counting and addition.

Although the data was quantified, emphasis in this report will be on a descriptive analysis of the children's strategies and the conclusions drawn from these. The evidence in this study clearly indicated that young children are able to solve a range of multiplication and division problems intuitively, in a condusive environment, such as a Problem-Centered classroom. This therefore suggests that children in the Junior Primary Phase should not be restricted to certain word problems in multiplication and division; they should be given the opportunity to solve a range of word problems, intuitively.

## REFERENCES

Mulligan, Joanne T.: 1992, Children's solutions to multiplication and division word problems: a longitudinal study, **Proceedings of the Sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education**. (2) 144-151. Durham, NW (USA).

Murray, H., Olivier, A., & Human, P.: 1992, The Development of young children's division strategies, **Proceedings of the Sixteenth International Conference for the Psychology of Mathematics Education**, (2) 152-159.

Murray, H., Olivier, A., Human, P.: 1993, Voluntary interaction groups for problem-centered learning, **Proceedings of the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education**, (2) 73-80. Tsukuba, Ibaraki, Japan.

Penchaliah, S.: 1997, Young Children's Intuitive Solution Strategies for Multiplication and Division Word Problems in a Problem-Centered Approach. Unpublished M.Ed. University of Durban-Westville.

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# A look at teachers' professional knowledge through the design of class activities

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Research about mathematics teachers emphasizes their role as problem solvers: those problems that appear in specific teaching situations. This role is influenced by their professional knowledge (Even *et al.*, 1996). In this study participated 7 couples of secondary mathematics teachers from 7 schools from Bogota. Its aim was to describe some aspects of the teachers' professional knowledge, in special their beliefs about the teaching and learning of mathematics.

Each couple of teachers was asked to deal with a typical students' error that could appear in one of the courses they were teaching. In order to deal with this pedagogical problem, they had to design, implement and evaluate a 3-hours sequence of class activities that they thought could help students overcome the difficulties that, from their perspective, originated the error. We collected information from their *a priori* reflections about the mathematical, learning and teaching dimensions underlying their designs and from the designs themselves. We applied a content analysis technique to these sources in order to obtain information about the teachers' beliefs concerning the teaching and learning of mathematics and about the teachers' pedagogical and mathematical content knowledge.

Most teachers' performance suggest that they believe that students learn a particular mathematical concept as a result of going, step by step, through a learning path that the teacher has to design and control. They also believe that, in the process, students should mainly observe and analyze what the teacher considers important. Teachers base their didactical designs on assumptions about what they imagine that students already know. They also rely strongly on the series of necessary knowledge that students supposedly master for learning a specific content. Sometimes, these assumptions are not always appropriate. On the other hand, teachers' designs evidence their deficiencies in respect to their mathematical knowledge. This is shown in the fact that they do not make and point out connections among concepts, and between concepts and procedures. Their deficient pedagogical knowledge is reflected in the fact that they do not induce students to build these connections.

These results show the need for professional development programs that question teachers' beliefs about the teaching and learning of mathematics, and emphasize the teachers' professional knowledge in order to improve their ability for solving didactical problems.

## References

Even, R., Tirosh, D. & Markovits, Z. (1996). Teacher subject matter knowledge and pedagogical content knowledge: research and development. In Puig, L. and Gutiérrez, A. (Eds.). *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, vol. 1. Valencia: Universidad de Valencia, pp. 119-134.

# MULTIPLE CONCEPTIONS OF POINT: ENTAILMENTS AND CONFLICTS

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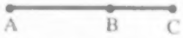
Consider the following problem: 

Does $\overline{AC}$ have more points, less points, or the same number of points as $\overline{AB}$?

This problem raises several interesting mathematical questions. How do we compare the cardinality of infinite sets? In what cases can a set be the same size as a proper subset? However, the difficulties people may have with this problem are not only related to issues of infinity, but also to the seemingly more basic issues of point. It turns out that some people's understanding of point prevents them from answering-- or even raising-- the interesting mathematical questions.

In the study reported here, I looked at how eleven adults responded to this problem. I compared those who had conceptual difficulties with this problem to those who did not. I found that a significant source of people's conceptual difficulties stemmed from their understanding of point. Some people did not appear to be aware that they had multiple conceptions of point and that the entailments of each conception could conflict. Those who did not have difficulty with this problem also had multiple conceptions of point. However, they were able to choose an appropriate conception for this problem and identify the relevant mathematical rules associated with that conception. These results suggest that it is important for people to realize that (1) they have multiple conceptions of a particular mathematical concept, (2) multiple conceptions can lead to conflicting mathematical claims, and (3) given a particular problem, certain conceptions are more appropriate than others.

This study connects to research on epistemology and metacognition. The problem falls into a broad and important class of problems involving epistemological obstacles (Sierpinska 1994). Moreover, the results demonstrate how metacognition can play a significant role in helping people resolve their conceptual conflicts (Schoenfeld 1992).

## References

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp. 334-370). New York, NY: Macmillan.

Sierpinska, A. (1994). *Understanding in Mathematics*. The Falmer Press.

# LEARNING GEOMETRY AS MEANS FOR INCREASING COGNITIVE RESOURCE

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We are trying to find a reply to the question: should geometry be taught at school and why? The answer depends on the set objectives. If the objective of education is formation and increase of cognitive resource [1], which enables a person, according to his cognitive style (thinking peculiarities) [4], to be successful [3], then geometry offers greater didactic possibilities than other parts of mathematics. These possibilities can be based on a child's psychological background, at the same time taking into account a learner's various thinking and imagination abilities. Although the latter, to a certain extent, are genetically determinated [6], much can be acquired in one's life [5]. As a result a person is directed to a greater field independence, which leads to further success in many specialities. In particular, by learning geometry, space imaginativeness can be trained. Generally, there are various recognized and unrecognized aims of geometric knowledge and its applications. For example, in the case of a learner the visualization of thought and its use for comprehension and retention are very important. People with artistic imagination live their valuable lives playing with emotions, lights and shades, and colours. In general they proceed from an object or arrive at an object which often is of a geometric kind. We even talk about geometric art (K.Malevic, R.Meel), which usually represents a real environment. However art also includes geometric curiosities (M.Escher, K.Põllu) which represent, for instance, the differential geometry of spaces with constant curvature. Thus besides designers, architects and engineers, who need a knowledge of descriptive geometry for their measurable drawings, scientists can present their newest theories graphically (e.g. theory of catastrophes) as well. This sphere of knowledge is said to be descriptive topology [2]. It essentially complements the possibilities of contemporary computer graphics.

Schools ought to provide learners with some everyday geometric knowledge and experience of space imagination with connection to logical thinking. At the same time, they would prepare a basis of geometrical knowledge for future fine artists, architects, engineers and scientists. Thus the obligatory curriculum of the aspiring teacher of mathematics should certainly include elementary, analytical, differential and descriptive geometry. Thereafter it may remain everybody's own decision whether or not to take some courses of contemporary geometry, which should widen one's view and permit an appreciation of geometry in everyday life. A teacher with such preparation will best be able to meet the needs of students with different learning styles and objectives in their pursuit of geometric knowledge.

**REFERENCES. 1.** Fiedler,F. (1978): *A Contingency Model of Leadership*. In: Group Processes. Edited by Berkowitz. Academic Press. pp. 165-208. **2.** Francis,G.K. (1988): *A Topological Picturebook*. Springer-Verlag. New York, Berlin, Heidelberg, Paris, Tokyo. **3.** Grasha,A. (1986): *Practical Applications of Psychology*. Scott, Foresman and Company. Glenview, Boston, London. **4.** Kelly,G.A. (1955): *The Psychology of Personal Constructs*. Vol.1. A Theory of Personality. New York: W.W.Norton & Co. Inc. 555 pp. **5.** Toomsk,V. (1994): *Learning Styles*. Summary. Universitas Scientiarum Socialium et Artis Educandi Tallinnensis. Tallinn. **6.** Witkin,H.A., Goodenough,D.R. (1982): *Cognitive Styles: Essence and Origins*. New York: International Universities Press Inc.

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# DO THE CONCEPTS OF DERIVATIVE AND MARGINAL COST MAKE SENSE FOR BUSINESS STUDENTS?

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This paper is reporting on an exploratory study to describe students graphical and intuitive understanding of the concept of the derivative. A questionnaire with items relating to visual interpretation of the derivative as the slope of the tangent line, was administered to a calculus class at a business college in Norway.

We find that one half of the 138 students have great difficulties in separating the value of the derivative function at a point from the value of the given function itself, qualifying results in Norman and Prichard (1994). The study is indicating that most students do not have the ability to reverse the path of development of the notion of derivative (Eisenberg 1992) and they are unable to make inferences on the behavior of a function based on information on the derivative function.

One major purpose of this study is to investigate if working within an economic model would have an influence on students visual interpretation of the concept of derivative. By using the same kind of questions now rephrased for cost functions or revenue functions, we find that the economic context seem to have only a minor impact on the pattern of answers. Further testing on the concept of marginal cost, indicate that a majority of the students have a strong tendency to mix questions relating to marginal cost with the cost function itself.

Based on our findings we would say that only a minority of the students seem to have developed a sense of marginal cost making this a workable concept in economic modeling. Vinner (1992) is commenting on the serious gap between students' performances and expectations of the educational system. The analysis of this paper indicate that teachers and professors in economics as well as in mathematics, have to pay attention to this gap.

References:

Eisenberg T. (1992). On the Development of a Sense for Functions. In Harel G. and Dubinsky E. (Eds.), *The concept of function,* MAA Notes Vol. 25

Norman F.A. and Prichard M.K. (1994). Cognitive Obstacles to the Learning of Calculus. In Kaput,J.J. and Dubinsky,E. (Eds.), *Research Issues in Undergraduate Mathematical Learning,* MAA Notes Vol. 33

Vinner S. (1992). The Function Concept as a Prototype for Problems in Mathematical Learning. In Harel G. and Dubinsky E. (Eds.), *The concept of function,* MAA Notes Vol. 25

# The development of mathematical meaning in a computer-based environment[*]

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The aim of the present study is to examine the processes of meaning making that occur within a dynamic geometry environment, by questioning the role of (a) the computer, as a mediator, (b) the mathematical experience, and (c) the social interactions, and their relationship with the development of mathematical meaning.

The participants involved in the empirical study were four students working in group and the teacher in 8th grade mathematics classroom. The methodology adopted has an interpretative nature.

The study's framework draws from activity theory in the line of Vygotsky and Leont'ev. Mathematics learning is seen as a situated phenomenon. As the school context plays a fundamental role, it is not possible to separate activity, people acting--and respective interactions--and the artifacts that mediate that action. All those dimensions are intrinsically interwoven. Mathematical objects are ambiguous in the school context: they permit multiple interpretations because they are related with previous students' understandings. This ambiguity is reduced when teacher and students negotiate mathematical meaning. It's the emergence of intersubjectivity. "The context and the singular meaning elaborate each other. From this ethnomethodological point of view, (...) the context of school mathematics is continually constituted" (Voigt, 1994, p. 285). So it does not make sense to talk about mathematical objects without talking about the activity with its potential of transformation. Winograd and Flores (1987) present Heidegger's analysis of understanding: "The interpreted and the interpreter do not exist independently: existence is interpretation, and interpretation is existence." (p. 31).

In this short oral communication I will present the key theoretical ideas and some of the research findings. I will attempt to highlight some aspects that seem to play a crucial role on the search for meaning: the social interactions at the classroom and the nature of the mathematical activity mediated by the computer.

## References

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics, 26,* 275-398.

Winograd, T., & Flores, F. (1987). *Understanding computers and cognition: A new foundation for design.* Norwood, NJ: Ablex.

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# TWO LETTERS IN ONE EQUATION: UNKNOWNS OR VARIABLES?

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According to our research on elementary algebra appropriation conditions at secondary school, we have tried to know how the conception[*] -elaborated by the students when working with equations with only one variable- which says the equation is a numerical equality and the letters are numbers to "be discovered" ( cf [4]), influences the understanding of the nature of a two variable linear equation. For this that, we have interviewed students who have already undergone the learning process of linear systems at school -15/16 years old youngsters-. First of all, the students are given a word problem which describes a relation between the prices of two different kind of objects. This relation is represented by **a** two variable linear equation and the students are to find possible values for these prices. Next, the students are to solve a two equation system -one of them has already been seen in the first part of the interview- so as to find out which relations they establish between the solutions of the system and the solutions of the equation. The aim of this communication is to expose some of the results obtained during the analysis of these interviews.

- All the students mantain that the equations related to the problem has only one solution, basically giving two different justifications. The first one supported on the **problem** itself, once a solution is found, the real situation stated by the statement is refered to: " It´s useless to go on looking for more values, since you´ve already got the prices". The second is supported on the **equation** as an object, the letters -probably as an extension of the conception for a variable- are conceived as already determined but unknown numbers. "You have two letters that would be your unknowns. Once you´ve solved the equation, you´ll have an amount for each of the letters. And that´s it".
- The students -properly- expound that a pair of numbers is the solution to a system if it verifies each of the equations that forms it, but **immediately after** they have solved a given system, they do not see that the solution they have got, is the solution of one of the equations when they are told to take this equation isolated from the system.

These results clearly show that the object " two variable equation" has not been understood by the students despite their having already been working with linear equations systems with two variables. The educational system proposes to deal with these systems in such a way that it is possible to successfully operate with them without taking into account as an objective the students' understanding of each of the equations. Besides the problems that are solved in the classroom are always modelled through systems with only one solution. The old conception thus seems to automatically shift to the object " two variable equation" so that a wrong conception is shaped and the notion of variable is left aside. Janvier, Charbonneau an Cotret had already pointed out in [3] that the conception of unknown would become an epistemological obstacle when trying to get the notion of variable. Our analysis seems to point out that the educational system would be contributing to consolidate this obstacle[*], endowing it of a didactic character.

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[*] The notion of obstacle is considered as G. Brousseau does [2]. And the one of local conception as M. Artigue does [1].

[1] Artigue,M. (1990) Epistémologie et Didactique. Recherches en Didactique des Mathematiques, 10 2 (241-286)

[2] Brousseau,G. (1989) Les obstacles épistémologiques et la didactique des mathématiques. Construction des Savoirs. Obstacles et conflits CIRADE. Agence d´ARC inc.

[3] Janvier,C. Charbonneau,L. Cotret,S. (1989) Obstacles épistémologiques à la notion de variable: perspectives historiques Construction des Savoirs. Obstacles et conflits. CIRADE. Agence d´ARC inc.

[4] Panizza,M. Sadovsky,P. Sessa,C. (1996) The first algebraic learning. The failure of success. Proceedings of the PME20.

# EXAMINATION OF TEACHERS' PROFESSIONAL DEVELOPMENT

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In this inquiry we examine how the participation of teachers in a professional development program influences their behaviors, actions and beliefs. In the last three years, the teachers have been engaged in this program at a public university in Brazil. In a two- year course they have to study, discuss and reflect about mathematics content, methodology for mathematics teaching and mathematics education as well as implementing and reporting teaching experiments in their classrooms (Nunes, 1993; Raymond & Santos, 1995; Santos & Nasser, 1995). Afterwards, they have been engaged in a collaborative project of development of action research (Raymond, 1994). Throughout this professional development program they were challenged to think about curricular changes and to try out in their classes ideas about problem solving, cooperative work, use of written language in mathematics explanations and development of arguments and proofs, alternative assessment, etc. All the planning, implementing, analyzing, and reporting of teaching experiments in this program challenged teachers to think if they want or not to implement teaching changes and engage themselves in a role of reflective practitioners. This whole process created situations for them to experience cognitive and emotional dissonance concerning their teaching practices and pedagogical mathematical knowledge (Raymond & Santos, 1995). In such cases they question their attitudes, beliefs and actions towards mathematics teaching, learning and assessment. Especially the trials of alternative assessment involving both quantitative and qualitative methods were crucial to their rethinking of math content approaches, students' learning processes and classroom environment and to the development of their awareness concerning their teaching practices. The opportunities for the teachers to use their own classrooms as experimental laboratories for implementing changes in mathematics teaching, for reflecting about the difficulties involved in these attempts in a new context and for deciding on-site how to manage pedagogical issues were also fundamental for them to reflect-in-action and to reflect about their actions. Excerpts from the work of some teachers followed by analysis of them will be presented. It will also be displayed instances of: a) teachers' realizations about their own teaching practice, b) teachers' attempts to investigate students' math perceptions after trials of teaching experiments, and c) investigator's perceptions of this process of professional development leading to the development of teacher autonomy, awareness and desire to implement change.

**References**

Nunes, T. (1993). Romper o pacto gerar ideias (Breaking the pact and generating ideas). *Revista do Geempa, n° 2*, 85-93.

Raymond, A. M. (1994). Collaborative action research in mathematics education: A tale of two teacher-researchers. *Proceedings of PME-NA 16, Vol. 2*, pp. 67-73. Baton, rouge, LA, USA.

Raymond, A., & Santos, V. (1995). Preservice elementary teachers and self-reflection: How innovation in mathematics teacher preparation can challenge mathematics beliefs. *Journal of Teacher Education, Vol. 46, n° 1*, 58-70.

Santos, V., & Nasser, L. (1995). Teachers' awareness of the process of change. *Proceedings of PME 19, Vol. 2*, pp. 186-193. Recife, PE, Brazil.

# THE THREE DIMENSIONS OF ERROR IN THE UNDERSTANDING OF ALGEBRAIC LANGUAGE

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Difficulties associated with algebraic language in Secondary Education is beyond all doubt. These difficulties lead to students´errors, these errors being caused in very different ways and then being reinforced into complex networks. From a teaching-learning perspective, it is useful to possess means to analyse these errors in order to determine their characteristics, understand the over their underlying knowledge and design tasks that help build up algebraic language in a more meaningful way.

In this study, we analize the causes of errors commited by 12-13 year-old students while acquiring algebraic language. We use the following analytical elements: specific test (pre-test and post-test), students´classroom productions and videotaped interviews (case studies).

We have detected errors caused by obstacles and errors caused by a lack of meaning. These last ones have two different origins, one related with difficulties associated with the complexity of the mathematical objects and with the processes of mathematical thinking, and the other related to the difficulties associated with the affective and emotional attitudes the students have regarding mathematics. In this study we have found three dimensions of error: the "obstacle" dimension, the "lack of meaning" dimension, and the "affective/emotional" dimension. The "obstacle" dimension is characterized by pattern-governed errors and are consistent within studens' ways of thinking. There are three different types of obstacle: epistemological, didactic and cognitive. The "lack of meaning" dimension is characterized by the errors occurring at the various stages of cognitive development and which arise in the representation systems of algebraic language. Different errors occur at the three following stages: semiotic, structural and autonomous. The "affective/emotional" dimension is characterized by negative and emotional attitudes towards algebraic language and is associated with anxiety and fear: anxiety to finish a task (lack of concentration, distractions, forgetfulness, etc.), and fear of failure or mistake (beliefs, blocks, etc.).

This study of the three dimensions (which are obviously inter-connected), implies that we should concentrate on helping students in their difficulties in understanding algebraic language through teaching-learning strategies that are meant to overcome obstacles, lend meaning to algebraic objects and foster a rational attitude towards algebra.

## References

KILPATRICK.; GÓMEZ, P. Y RICO, L. (1994) *Educación Matemática*. Grupo Editorial Iberoamérica. México.

# CORRELATES OF STUDENTS' PERFORMANCE IN STATISTICS

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The education literature overwhelmingly conveys the value of statistics in related to research in teaching and learning at all levels of students. As such, teaching faculty in all disciplines and at all levels of education shared a common concern for students' performance in educational statistics. A review of education literature reveals several factors that have been found to influence performance in general. In a comprehensive literature review, Torres (1993) ascertained that five major factors that contributed to development of mental operations or cognitive abilities: 1) teacher-related variables; 2) student-related variables; 3) personal characteristics; 4) learning styles and 5) other factors. This study sought to investigate the factors related to performance in educational statistics of students enrolled in the Faculty of Educational Studies. A random sample of second and third year students was selected for the study. The Group Embedded Figures Test was used to gather data on students' learning style (Witkins, 1977). A questionnaire covering general study habits, mathematical background and personological data was also administered to the students. Using multiple linear regression, 9 percent of variance in students' performance was uniquely accounted for by learning style after controlling personal characteristics. Additional analysis showed that learning style was found to be correlated to study habits and interest in the course as rated by the students. Recommendations will be offered based on the results and for future research.

## References

Torres, R. M. (1993). The cognitive ability and learning style of students enrolled in the College of Agriculture at the Ohio University. Unpublished doctoral dissertation, The Ohio State University, Columbus.

Witkin, H.A., Moore, C. A., Goodenough, D. R., & Cox, P.W. (1977). Field-dependent and field-independent cognitive styles and their educational implications. Review of Educational Research, 47 (1), 1-64.

# School structure influence on administrators' actions upon mathematics staff development in schools[*]

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Recent studies in mathematics education have reported about the importance of involving administrators in the processes of school mathematics change (Peluso *et al.*, 1996). Administrators, specially the principal and the head of the mathematics department, are pivotal in the generation of a professional interaction among the mathematics teachers (Perry *et al.*, 1996). This study analyzed how some school structural conditions influence the leadership role that both the principal and the head of the mathematics department play in the promotion of professional interaction among the staff of mathematics teachers.

As part of the study, we proposed and induced the administrators from 15 schools in Bogota, to carry out small-scale action research projects in which they tried to give a solution to a particular problem related to the functioning of mathematics teaching in their schools. The interaction between us, the school administrators and some mathematics teachers was build around these activities. We obtained information using questionnaires and clinical interviews with the administrators. As a means of contrast, we also collected, through interviews, some teachers' perceptions about the administrators and the school mathematics staff. We conducted some case-studies in three of the schools in order to analyze in detail the relationships between the identified elements.

We found that there is a strong relationship between some school structural conditions such as the stability of the school structure given by the school size, the staff movility and the type of school –private or public– and the leadership role that the principal adopts in opening spaces for staff professional interaction, through empowering the head's leadership among the mathematics teachers. As a result of this empowerment, the mathematics department head is able to adopt an academic leading role and to propose staff development activities that overpass his typical function as administrative channel of communication between the principal and the teachers. It was also found that when there is a disturbance in the school structure stability the connection between administrators and professional interaction gets broken. Besides, the inexistence –in some schools– of a formal post of "head of the mathematics department" difficults the creation of an institutional coherence between the principal, the person in charge of the mathematics staff and the group of teachers, which favors professional interaction.These findings suggest that implementing changes for altering the teaching of mathematics through improving teachers' professional interaction needs to seriously consider the institutional complexities involved.

## References

Peluso, T. et al. (1996). Extending the educational conversation: administrator's views of staff development. In Puig, L. and Gutiérrez, A (Eds.). *Proceedings of the 20th. Conference of the International Group for the PME*. Valencia: University of Valencia, pp. 115-122.

Perry et al. (1996). The teaching of mathematics from within the school. Teachers and principals as researchers. In Puig, L. and Gutiérrez, A. (Eds.). *Proceedings of the 20th. Conference of the International Group for the PME*. Valencia: University of Valencia, pp. 123-130.

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# INVESTIGATING THE NATURE OF STUDENTS' UNDERSTANDING OF DEFINITE INTEGRAL: WALLIS OR RIEMANN'S IDEA

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This study explores the nature of calculus students' understanding of the definite integral. Traditionally in calculus courses in the USA the definite integral is interpreted and taught using the Riemann sum approach.

We analyzed the transcripts of 36 calculus students in which they interpreted the definite integral given in symbolic, velocity-distance and mass-density problem situation. The results of the analysis indicate that students' intuition does not necessarily follow the Riemann sum idea. There was some evidence of a strong presence of Wallis's idea when interpreting a definite integral as the area under the curve (in 2D for example). That is, a significant number of students interpreted the definite integral as the area which is the sum of an infinite number of parallel line segments whose length is the value of the function at a point. Smaller number of students interpreted the definite integral using only Riemann sum idea while others went back and forth from Wallis to Riemann sum ideas.

Our finding raises the question of its pedagogical implication. Should we teach the concept of definite integral from the Riemann sum approach only when it seems that students' intuition naturally goes to Wallis idea? Dennis and Confrey (1996) suggested that we as teachers and researchers should use students' intuitions, inventions and resources to improve their understanding of mathematical ideas.

We will further discuss our findings, the pedagogical implications and suggestions for further research at the conference.

Dennis, D., & Confrey, J. (1996). The creation of continuous exponents: A study of the methods and epistemology of John Wallis., *Research in Collegiate Mathematics Education II*, pp. 33-60.

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# COMPUTERS, GRAPHICS AND REFUTATIONS[1]

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This work aims to present an analysis of an episode from a one-hour-and-a half-interview with two students, Vitor (21 years old) and Ricardo (18 years old) while working with a mathematical question in a computational environment using the software *Derive*. These students were attending a first-year mathematics course for Biology majors at the State University of São Paulo (UNESP) at Rio Claro, Brazil. In this course graphic calculators were used.

This interview was the first of a set of three interviews that were made by the author, entire'y videotaped and then transcribed. After a first analysis of the data the episode was selected because it showed some important aspects of the students' thinking.

Based on the analysis a model of the students'thinking processes was built. This model will be contrasted with the model Davis & Hersh (1988) elaborated synthesizing Lakatos' proofs and refutations heuristic.

In *Proofs and Refutations*, Lakatos (1976) challenges the formalist bastion, the mathematical dogmatism, the immutability of mathematics and elaborate "*the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase of the numbers of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations*" (1976, pp. 5).

The students' strategies in a computational environment, while solving the problem posed, are characterized by a *zigzag* of conjectures that are barring and then reformulated. Trial and error, intuition, induction, conjectures and refutations are elements appearing in the students' work. These elements also appear in Lakatos' description of the logic of mathematical discovery: from a naive conjecture, it follows a set of counterexamples, refutations and reformulations leading to new conjectures and proofs. It is in this process of conjectures, challenges, proofs and refutations that mathematics is produced by human beings.

The educational consequences of Lakatos' Philosophy has been discussed inside the Mathematics Education Community (Hanna, 1996; Garnica, 1995; Ernest, 1991) and exerts a usually non critic fascination, leading to extrapolations and transpositions without considering some aspects. In this study Lakatos' work is considered just as a way of seeing the mathematicians' thinking which will be contrasted with the model of the students' thinking.

**References**

Davis, P. & Hersh (1988) *Experiencia Matemática*. Editorial Labor, Madrid.
Ernest, P. (1991) *The Philosophy of Mathematics Education*. The Falmer Press.
Garnica, V. (in press) *Lakatos e a Filosofia de Provas e Refutações: considerações*. Rio Claro.
Hanna, G. (1996) The ongoing value of proof, in Puig, L. & Gutiérrez, A. (Ed.) *Proceedings of the 20 th PME Conference*, Valencia, vol 1, pp. 21-34.
Lakatos, I. (1976). *Proofs and Refutations*. Cambridge University Press.

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[1] This research was developed within the activities of the Research Group *Technology and Mathematics Education* (*Informática, outras mídias e Educação Matemática*), chaired by Marcelo de Carvalho Borba, from The State University of São Paulo (UNESP), Rio Claro - BRAZIL. I have had the opportunity to discuss some of these issues with Ole Skovsmose, Marcelo Borba and Miriam Godoy P. da Silva. I am grateful for their insightful comments.

# A FIBONACCI GENERALIZATION: A LAKATOSIAN EXAMPLE

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A generalization of the Fibonacci series which roughly followed a Lakatosian heuristic will be presented. It was started by the accidental discovery of a Grade 9 class of a gifted South African mathematics teacher. Basically, the discovery was the following: If we call the *n*th term of a Fibonacci series $T_n$ and its sum to $n$ terms $S_n$, then $T_n + T_{n+k} = T_{n+k+1} \Leftrightarrow 1 + S_n = T_{n+k+1}$.

Kendal, one of his Grade 11 pupils, then came up with what appeared to be a convincing proof and which was later published in a South African mathematics education journal (De Jager, 1990). However, closer examination by myself through testing specific cases revealed counter-examples to both the forward and converse statements. These were subsequently presented as a challenge to the readers of that journal to see whether "Kendal's theorem" could be saved (De Villiers, 1990). This led to further generalizations and a reformulation of proofs, as well as a necessary clarification of the normally meaningless phrase $S_0$ (Schutte, 1991; Du Toit, 1991; De Villiers, 1991).

This example clearly reveals the sometimes important interplay between quasi-empirical investigation and logical analysis in the production of mathematical knowledge.

## References

De Jager, C.J. (1990). When should we use pattern? **Pythagoras**, 23 (July), 11-14.

De Villiers, M. (1990). A counter-example to Kendal's theorem. **Pythagoras**, 24 (Nov), 5.

De Villiers, M. (1991). The meaning of $S_0$ and a further generalization of Kendal's theorem. **Pythagoras**, 27 (Nov), 7-8.

Du Toit, K. (1991). Kendal se Stelling en Bewys: Wat gered kan word. **Pythagoras**, 25 (April), 5-6.

Schutte, H.J. (1991). No substitute for logical reasoning. **Pythagoras**, 25 (April), 5.

# LOOKING THROUGH THE GRAPHICAL CALCULATOR: AN EXAMINATION OF BELIEFS CONCERNING THE TEACHING AND LEARNING OF ALGEBRA

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This report is a summary of one aspect of research into the ways in which graphical calculators mediate both mathematical and meta-mathematical activity. The report uses evidence gathered through a three year association with three London comprehensive schools and a recent intensive project working with girls aged 13+ in one of those schools.

The adoption of a Vygotskian perspective, in particular Vygotskian notions of tool use (Kozulin 1990), has allowed me to draw inferences from the observations I have made during the project. From this perspective, I will report, very briefly, on my observations of how the calculator was used as tool for learning and teaching ideas of function and graphs (Winbourne 1992). My main concern, however, will be to illustrate how the calculator appeared to function as a tool for enabling students and teachers to articulate and challenge their beliefs about the nature of the activity in which they were engaged. The calculator provided a window in the sense described by Noss and Hoyles (1996) on the ways in which mathematical meanings were made, but it also apeared to do more than this. At a meta level, it enabled students, teachers (and this observer!) to see this activity as socially situated in communities of practice (Lave 1991) some of whose central characteristics, thrown into relief by the use of the technology, could then be identified.

Vignettes will be used to provide a basis for discussion and questions: through one of these I will suggest that through her use of the calculator to explore polar equations, Sarah (aged 13) enabled all who were in the classroom with her in some way to begin to question how 'linear' relationships might be represented; through another, I will show how the power afforded to the users of graphical calculators enables them to see the mathematical practices into which they are being inducted as a socially situated selection of those that might actually be available to them.

Kozulin, A., 1990, Vygotsky's Psychology: A Biography of Ideas, Harvester Wheatsheaf, London.

Lave, J & Wenger, E, 1991, Situated Learning: Legitimate Peripheral Participation Practice, Cambridge University Press, New York.

Noss, R. & Hoyles, C., 1996, Windows on Meanings: Learning Cultures and Computers, Kluwer, Dordrecht.

Winbourne, P., 1992, Calculator Revision, Micromath, Vol. 8, No. 2, Association of Teachers of Mathematics

# IZA'S CONCEPTIONS OF DECIMALS

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Studies on the teaching and learning of decimals had focused on, among others, analizing error patterns on problems involving decimals, studying misconceptions that students have on the decimal concept and operations involving decimals, and studying the difference between solutions produced by novices and experts (Bell, Swan, & Taylor, 1981; Graeber & Tirosh, 1990; Markovits & Sowder, 1991; Nesher & Peled, 1986). Many of these studies had revolved around the computational performance of students rather than on the basic concepts of decimals. In this paper, Iza's conception of decimals will be discussed. This report is part of a study on the decimal schemes of primary school students and how students use their decimal schemes in solving problems on decimals. The study was conducted within the constructivists' framework. Data incorporating both verbal and non-verbal behaviour was gathered from eight problematic activities administered in five successive clinical interview sessions. Each session was within 30 minutes to an hour and all sessions were videotaped. The analysis of data involved transcriptions of interviews into written form, development of case studies describing each subject's behaviour on certain aspects of decimals, and identification of each subject's behaviour patterns. Iza's conception of decimals is reported from the child's perspective based on the author's interpretations of the subject's conceptions of decimals. Iza viewed decimals as another form of fraction with denominators not necessarily in powers of ten. The numerator was determined by the size of the 'number' after the decimal point whilst the denominator was any number greater than the numerator. The paper also discusses the classroom experiences that may have influenced Iza's conceptions of decimals.

## References

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. Educational Studies in Mathematics, 12, 399-420.

Markovits, Z. & Sowder, J. T. (1991). Students' understanding of the relationship between fractions and decimals. Focus on Learning Problems in Mathematics, 13(1), 3-11.

Nesher, P. & Peled, I. (1986). Shifts in reasoning. Educational Studies in Mathematics, 17(1), 67-80.

Graeber, A. O. & Tirosh, D. (1990). Insights fourth and fifth graders bring to multipliction and division with decimals. Educational Studies in mathematics, 21, 565-588.

# Undergraduate Mathematics Education: Towards an innovative classroom practice

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Modular approach has been reported to provide better learning opportunity compared to conventional instructional methods (Gahlot, 1996). Modules are self-paced and self-instructional learning resource. What is more important, it recognizes the nature of individual differences among the learners. Therefore, we consider it as a promising innovative method in mathematics teaching at UTM that has students of wider range of abilities. This paper reports some preliminary results on an on-going research investigating the possibility of learning beginning calculus through modules. The design and the development of the modular materials are based on Rex Meyer's (1984) approach together with Mason, Burton and Stacey's (1982) technique of problem-solving. To validate the module, the first draft was tried out with a group of peers, a small group of target students and a sample of a class of students. Here, we only present the outcome of the trial out with our sample class–a class of 50 first year Civil Engineering undergraduates with mathematical background ranging from the 90th to 40th percentile. Data collected give indication of the appropriateness and effectiveness of the module. In particular, there was more active involvement of the students and the learning activities had led to the achievement of the objectives. However, the data also shows some necessary changes and rewriting of the module draft.

## Reference

1. Ghalot, P.S. (1996). *Module Writing – An Overview.* Paper presented at Special Workshop on Module Writing (phase III), Teaching Learning Centre, UTM Johor Bahru, Malaysia. 17 – 23 November.

2. Mason, J., Burton, L. & Stacey, K. (1982). *Thinking Mathematically.* London: Addison Wesley.

3. Rex Meyer, G. (1984). *Modules from Design to Implementation.* Colombo Plan Staff College for Technician Education.

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# Teachers' and Students' Conceptions of Mathematics and Mathematics Instruction

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1. Some *goals* and *reasons* for this study:

   - Empirical: It should be checked, whether several trends in mathematics education (found in the literature) could also be determined in classrooms.
   - Practical: One has to determine, understand and respect the "mental home" of students and teachers before reaching out for change - normally understood as "improvement" - and new common goals. Knowledge of individual differences and different profiles of beliefs might be especially useful to find individualized approaches for further developments. In this respect comparative studies are also of great value (as carried out and proved, e. g., by E. Pehkonen in cooperation with colleagues from many other countries).

     Results of such research might help teachers to become aware of and reflect on their beliefs about learning and teaching of mathematics. Such metacognitive activities might be a starting point for striving for change and improvement.
   - Theoretical: Beliefs are an important element of control in the mathematical problemsolving processes of students as well as in the teaching processes of teachers. More differentiated information from this field of research might help to improve theories of learning and teaching of mathematics.

2. *Methods*: Questionnaires were administered to 107 teachers and 2658 students - grade 7 mainly - from different types of school in the area of Hamburg (no random samples!). Some of these students were taught by some of those teachers.

   Statistical methods (SPSS; nonparametric tests mainly, esp. "Quickcluster") were applied as heuristical tools mainly to get information on possible preferences, individual differences (with respect, e. g., to gender and type of school) and different profiles (clusters) of students' and teachers' beliefs.

3. *Some results*: a) Teachers: They took methods of visualization in mathematics instruction to be most important. There was placed also strong emphasis upon heuristics and connections within mathematics and to other fields. Less was placed upon history of and esthetics as well as use of computers in their math-classes. The following clusters were determined: "New Math cluster" (representing the majority of our participants), "problem solving cluster", "advocates of social aspects/pupils-orientation/lower achievers", "advocates of pragmatism and discipline", "advocates of an elementarizing teaching style".

   b) Students: Understandability is most important for our students. Teachers are expected to explain as much and as good as possible. There was no one-to-one correspondence between teacher and student clusters. Examples: "cluster with strong teacher orientation", "c. oriented towards learning-schemes" etc.

## POSTERS

# SOME MISCONCEPTIONS ABOUT AVERAGES IN PROSPECTIVE PRIMARY SCHOOL TEACHERS

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In spite of the apparent simplicity of averages, many researchers have described difficulties in its understanding by students at different educational levels (e.g., Gatusso and Mary, 1994). In this work we present an assessment of these difficulties for future primary teachers, with the aim of adequately guiding the teaching of this topic. The data was taken from a sample provided by 132 prospective teachers, who completed a written questionnaire taken from Konold and Garfield (1993). Additionally we interviewed some of these students to clarify their conceptual errors.

The analysis of the answers shows that these future teachers have difficulties in understanding the following points: Dealing with zero and atypical values when computing averages, relative position of mean, median and mode in asymmetrical distributions, choosing an adequate measure of central value and using averages to compare distributions.

We conclude that the traditional approach to studying central position values, which is based on algorithmic definitions and on computing averages in decontextualized data collections, does not allow pupils to fully understand the meaning of the concept, what must include the following: a) relationships of averages with other central position values; b) representativeness of mean in symmetrical distributions; b) the mean as expected value in random sampling processes; c) the mean as fair quantity to distribute for obtaining uniform distributions in finite populations.

## References

Konold, C. & Garfield, J. (1993). Statistical reasoning assessment, Part 1: Intuitive thinking. University of Massachusetts: Scientific Reasoning Institute.

Gattuso, L., & Mary, C. (1996). Development of concepts of the arithmetic average from high school to university. In L. Puig y A. Gutiérrez (Eds.), *Proceedings of the XX Conference on the Psychology of Mathematics Education* (v2, pp. 410-408). University of Valencia.

# PRE-SERVICE TEACHERS AND THE GEOMETER'S SKETCHPAD

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Some pre-service mathematics teachers reveal negative attitudes and difficulties when they study theoretically geometrical concepts and relations, without the resource of a interactive geometry software.

So, the aim of this study is to analyse the performance and the attitudes of pre-service teachers when they work on geometric tasks using the Geometer's Sketchpad (GSP).

The main questions of the study are: (1) What attitudes reveal pre-service teachers when they use GSP to study geometry?; (2) What processes that underlie mathematical thought they use?; (3) How could we characterise the performance of the pre-service teachers?

The participants in this study are a whole class of the third year of the Maths and Science course in a School oh Higher Education, during 12 weeks. The participants worked always in pairs with one computer. None of the participants had knowledge of GSP. However they already work with computers and applications like Word and Excel. According to the aim of the study, I decided to focus on a qualitative methodology.

Data has been collected through geometric problem solving, observations and interviews. Data analysis was holistic, descriptive and interpretative. Principal results allow us to say that a change of attitudes of the pre-service teachers may have occur. They become more motivated, reflective and they self-regulate better their own work. They use more easily processes that underlie mathematical thought, such as conjecturing and generalising. Their performance was in a middle grade, but some of them have a very good performance.

## References

Labord, C. (1993). The computer as part of the learning environment: The case of Geometry. In C. Keitel & K. Ruthven (Eds.), *Learning from Computers: Mathematics Education and Technology*. Berlin: Springer-Verlag.

Mason, J. (1992). Researching problem solving from the inside. In J. Ponte et al. (Eds.), *Mathematical problem solving and new information technologies*. Berlin: Springer-Verlag.

# MATHEMATICS WHICH IS SURPRISINGLY CHALLENGING TO TEACH IN THE EARLY YEARS OF SCHOOLING

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Serious mathematics books in primary teacher education tend to be of two types: those which focus almost exclusively on the research literature and those which explain mathematical concepts for the intending teacher. The majority of students who find time to read tend to opt for the latter variety. Typically, even the most successful of these (e.g. Haylock, 1996; Thyer and Maggs, 1991), are fairly single-minded in their endeavour to explain mathematical concepts and procedures. Rarely, if ever, do such books make any distinction between topics which are straight-forward to teach and those which appear to cause considerable difficulty for teachers and pupils alike.

This poster session will illustrate and examine the research findings of a project involving 12 elementary school teachers, their pupils and the research literature. The focus will be on mathematical topics which the teachers have found particularly difficult to teach (e.g. place value and time). It will describe how the group endeavoured to develop their teaching skills through the sharing of:

(1) an underlying understanding of the complexities involved in the learning of some mathematical concepts. (See, for example, the work of Kamii, 1985, 1989, 1994 and Anghileri, 1995)
(2) teaching techniques used in other countries (e.g. there are wide variations in the way place value is taught)
(3) group member's own practical successes when working with concepts which others find difficult to teach.

Some suggestions as to effective ways to teach surprisingly challenging concepts will be given.

## References

Anghileri, J. (Ed., 1995) *Children's Mathematical Thinking in the Primary Years* London: Cassell

Haylock, D.W. (1996) *Mathematics Explained for Primary Teachers* London: Paul Chapman Publishing

Kamii, C.K. (1985, 1989, 1994) *Young Children (Continue) to Reinvent Arithmetic* New York: Teachers' College Press

Thyer, D. And Maggs, J. (1991) *Teaching Mathematics to Young Children* London: Cassell

# GRAPH FOR VISUALIZING TRANSITIONS (GVT) IN SEQUENCES OF PROBLEMS WITH SEVERAL QUESTIONS

Juan Antonio Garcia-Cruz and Antonio Martinón
Universidad de La Laguna

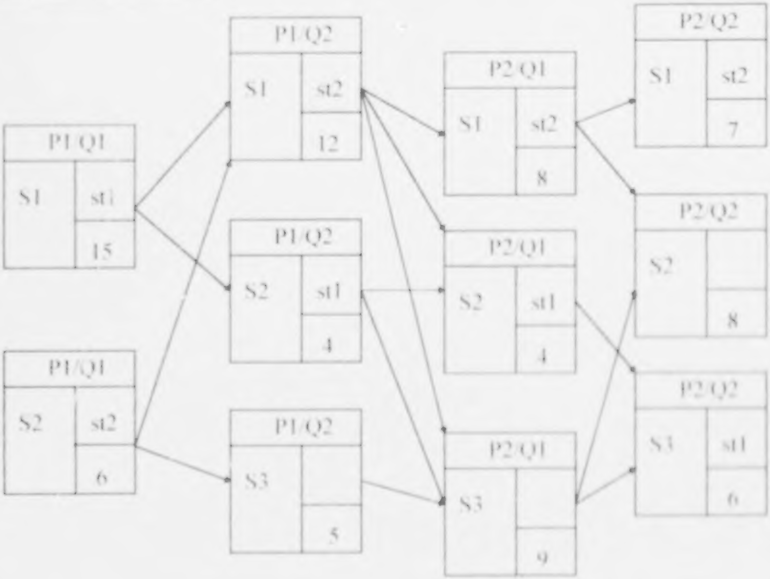
During our research project on problem solving strategies used by secondary students, we had developed a visual tool which allows us to present the information concerning the strategy used by a student or the whole group, the specific question within a problem and transitions between problems.

An abstract example is given below. The situation corresponds with a sequence of two problems posed to a group of 21 students (st1 and st2 are two students). There are three different strategies (S1 to S3) and each problem consists of two questions given in sequence.

The visual tool can display the information in a comprehensive way, so you can follow up the transitions from problem to problem through questions, with no need of doble-entry-tables any more.

(We also have a simplified version, we don't like it, and the margin is too narrow to contain it)

| Problem / Question | |
|---|---|
| Strategy<br>or<br>Calculation | Student<br>Group<br>Frequency |


P1/Q1
S1
st1
15
P1/Q1
S2
st2
6
P1/Q2
S1
st2
12
P1/Q2
S2
st1
4
P1/Q2
S3
5
P2/Q1
S1
st2
8
P2/Q1
S2
st1
4
P2/Q1
S3
9
P2/Q2
S1
st2
7
P2/Q2
S2
8
P2/Q2
S3
st1
6

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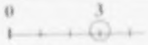
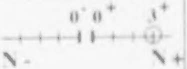
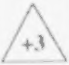
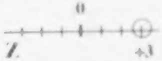
# NUMBERS AND MEASUREMENTS IN THE ADDITIVE CONCEPTUAL FIELD

González Marí, J. L., Dpt. of Didactics of Mathematics, University of Málaga

| formal domain | conceptual field (1) | measurements | numbers & signs | relations & compositions | representations & structures |
|---|---|---|---|---|---|
| (N, +, ≤) | A<br>natural | Cardinal, Numerosity:<br>"3 pesetas"<br>"3 marks at thermometer" | | combination: internal composition law (natural addition) |  <br>(N, +, ≤)<br>additive ordered semigroup |
|  | B<br>relative natural | Relative natural (directed) (2) measurements:<br>"3 ptas. more"<br>"3 marks more"<br>"a rise of 3 °C"<br>"a profit of 3 pesetas" | | comparison transformation - specific composition law: annulment-compensation and natural addition |  <br>non conmutative additive semigroup with a partial order and two zeros |
| (Z, +, ≤) | C<br>integers | startless and endless scales:<br>"a balance of +3 pesetas"<br>"a temperature of +3 °C" |  | combination; internal composition law (integers addition) (*) |  <br>additive ordered group |

### Additive and order structural differences between B and C conceptual fields (3)

| B).- Relative Natural | C).- Integers |
|---|---|
| a).- A partial order with inversion | a).- A total order |
| b).- Two ordered series with first element | b).- Without first element |
| c).- "Discontinuity" when crossing by zero | c).- "Continuity" when crossing by zero |
| d).- Double zero (no neutral element) | d).- Unique zero (neutral element) |
| e).- Natural add & annulment-compensation | e).- Integers addition |

(3) Stated from a didactic analysis and a research on cognitive differences (ref. 3)

(*) intensive magnitudes excluded

## References

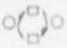
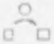
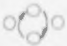
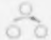
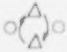
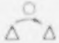
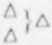
(1) Vergnaud, G.; Durand, C. (1976). Structures additives et complexité psychogénétique. *Revue française de Pédagogie*, 36, pp. 28-43.

(2) Bell, A. (1982). Looking at children and directed numbers. *Mathematics Teaching*, 100, pp. 66-72.

(3) González J. L. (1995). *El campo conceptual de los números naturales relativos* (Tesis Doctoral). SPICUM Universidad de Málaga.

(4) Carpenter, T.; Moser, J.; Romberg, T. (ed.) (1982). *Addition and subtraction: A cognitive perspective*. Hillsdale, New Jersey: LEA Publ.

### Schemes of simple additive situations (4) according to their semantic structures

|  | comparison | transformation | combination |
|---|---|---|---|
| natural |  |  | |
| relative |  |  | |
| integer |  |  |  |

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# CATEGORIES OF STUDENTS ACCORDING TO THEIR BELIEFS TOWARDS MATHEMATICS AND MATHEMATICAL PROBLEM SOLVING

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*This paper reports on a study that aimed to investigate the ways in which the students' beliefs about themselves as doers and learners of mathematics and about mathematics and mathematical problem solving were influenced by the instruction focusing on teaching mathematics via problem solving and based on metacognitive strategies. Despite individual differences, the students appeared to be falling into three distinct categories. Each category consists of the students who shared similar beliefs. Three categories of students ranged from resisting change to accepting change to embracing change. The three names, "traditionalists," "incrementalists," and "innovators" have been chosen based on the common characteristics, as labels for the categories.*

A review of the literature suggests a number of important factors regarding teaching and learning of mathematical problem solving. These factors are categorized by Lester, Garofalo and Kroll (1989a; 1989b) as knowledge, control, affect, beliefs, and socio-cultural factors. In order to make sense of what students do when they are engaged in problem solving activities, Schoenfeld (1987a; 1987b) suggests that it would be helpful to look at the students' cognitive resources, their knowledge of problem-solving strategies, metacognitive or managerial skills, and belief systems. Traditional classrooms with teachers lecturing most of the time and students passively taking notes are not flexible enough to allow an inquiry of that kind. Therefore, creating an appropriate learning environment with a problem-solving approach seems to be essential for allowing such detailed investigation. This paper is drawn upon the study that was conducted in a regular mathematics classroom and in which the teacher/researcher was limited by the same constraints as classroom teachers are; including certain deadlines, requirements, students' number in class and forty persent of the total grade for the final exam. Therefore, the findings of the study could be useful for practicing teachers and giving them some ideas about the ways in which students' beliefs about mathematics and mathematical problem solving might be influenced by the instruction students recieve in their own classrooms.

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# STUDY OF THE PREPARATION STAGE IN THE RESOLUTION OF BADLY DEFINED PROBLEMS

Noda Herrera, M. A., **Hernández Domínguez J.** y Socas Robayna, M. M
Universidad de La Laguna

From the definition of problem that from artificial intelligence Banerji (1980) proposes we characterized problems as well or badly defined. Taking them as a reference we did a research Project that intended to analyze the need to incorporate badly defined cientific and mathematical projects to school curricula as well as to study what happens in the resolutor when such problems are proposed. The research project is centred in the study of the resolutor's behaviour in the preparation stage (Bourne and others 1979) with well and badly defined mathematical problems in different contexts (arithmetic, algebraic and geometrical) and analyzing the cognitive representation systems that take place (visual, formal, strategies, etc.). We did the experience with 17 last year students of the title of Teacher specialised in infant schooling at the Faculty of Education of the University of La Laguna (Tenerife, Spain). These students are characterized by their low level in mathematical knowledge and reasoning, because they have not taken mathematics in their last years at their high school. They are divided into 3 groups and they have to resolve a battery of test (containing well and badly defined problems) in two one hour sessions in different days and with different instructions in each session. The first instruction is: "Solve the following problem. If you can't, explain why". And the second is: Solve the following problem. If you can't because it is badly set out, explain why it is badly set out, set it out properly and solve it". Group 1 (13 students) did the tests individually and in written form. Group 2 (2 students) solvelthem individually and with an interview. Group 3 (2 students) solvelthem with a discussion and debate among them. Groups 2 and 3 were videotaped. To study the resolutor's performance we have established a category system which studies the students' performance in relation to the series of well and badly defined problems, to the different contexts as well as to the different ways of organizing the information, the relevance or irrelevance of the information they use, the different representation system and the similarity and differences which appear in the resolutors. Moreover, we asked groups 2 and 3 questions, such as: Should Mathematics set these hinds of problems? Would you use them whit your pupils? How would you do it? How important is that pupils exercise with these hind of problems? Would parents accept those hind of problems? And the pupils? With the results we observe that the inclusion of "badly defined" problems enriches the preparation stage in the resolution of a problem and it facilitates the researcher the analysis and interpretation that the resolutor does with the given information, the restriction and modification of the information and the resolution criteria. On the other hand, the presence of well and badly defined problems seems not to create confusion among the these students. On the contrary, it seems to favour the resolution of the well defined problems, as it improves the presentation stage. Moreover, their behaviour in relation to the different types of badly defined problems suggests different ways of acting.

**References:**

BANERJI, R. B., 1980. *Artificial Intelligence. A Theoretical Approach.* (Elsevier North Holland, inc. New York), pp. 193

BOURNE, L. E., DOMINOWSKI, R. L. y LOFTUS, E. F., 1979. *Cognitive Processes.* Prentice-Hall. New Jersey

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# MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE AND THEIR PROFESSIONAL DEVELOPMENT

Susanne Hohoff
University of Duisburg, Germany

The professional development process by which teachers change their classroom practices, their knowledge and beliefs about learning and teaching mathematics as well as their role as teachers is fundamentally a learning process. The identification of teacher change with a learning process has been modelled as „teacher professional growth" (Clarke & Peter, 1993). The knowledge about the nature of this learning process itself and specially about the pedagogical content knowledge is still incomplete. Shulman (1986) has divided professional knowledge of teachers into „content knowledge", „curricular knowledge", „pedagogical knowledge" and „pedagogical content knowledge". Pedagogical content knowledge contains pedagogical and mathematical knowledge. He (1987, 8) describes pedagogical content knowledge as „... that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding." Teachers have acquired theoretical scientific knowledge about pedagogy and mathematics during teacher training. But we should not forget the implicit knowledge, the beliefs and conceptions of mathematical teachers about learning mathematics as well as their own knowledge and beliefs they have got in school. Teachers adapt or even change their individually constructed pedagogical content knowlege also in classroom interactions. Therefore we must separate theoretical didactical (mathematical) concepts from individual pedagogical content knowledge.

We will provide the opportunity for teachers to construct new understandings about teaching and learning, the roles they assume and the nature of change in a dialogue community. Because of the social constructivist principles of this study we take the professional classroom practice not only as a starting point in this project and investigate the development of pedagogical content knowledge during the whole time of the study.

Bromme, Rainer (1995). Was ist "pedagogical content knowledge"? Kritische Anmerkungen zu einem fruchtbaren Forschungsprogramm. In: Zeitschrift für Pädagogik, 33. Beiheft, 105-114

Clarke, D.J., Peter, A. (1993). Modelling teacher change. In: Atweh, B., Kanes, C., Carss, M. (eds.) Contexts in mathematics education. Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia. Brisbane: MERGA, 167-175

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. In: Harvard Educational Review, 57, 1-22

Shulman, L. (1986). Those who understand: knowledge growth in teaching. In: Educational Researcher, 15, 4-14

# variables and parameters of linear and quadratic functions

Bat-Sheva Ilany

Beit Berl Teacher Training College, Israel

The use of variables and parameters lies at the foundation of algebra. Dealing with these variables and parameters is the basis of understanding algebra and analysis. The great importance of signs and symbols in mathematics in general, and in algebra in particular. For instance, algebraic equations that contain letters represent not only themselves but an entire group of equations that have an identical construction (e.g., functions).

The linear and quadratic functions play a major role in the study of mathematics in Junior-high school and at high school level. These functions are formulated by variables and parameters. I have, therefore, chosen to focus on the use of letters in general and linear functions in particular, as perceived by pre-service teachers and advance mathematical students.

Subjects were given identical questions referring to above two types of functions (i.e. linear and quadratic). The research indicates the following results.

The roles of the parameters is generally clear to the majority of the subjects, with regard to the formal linear equation $y=ax+b$.

Concerning the formal quadratic equation $y=ax^2+bx+c$, the subjects performances are poorer. Parameter "a" is a coefficient which determines the change rate of the slope of the tangent to the curve. In other words: It determines the curvature of the function. The "a" sign, shows the sense of the curve (whether it is convex or concave).

Most of the examinees attribute the determinor of maximum / minimum of the parabola to parameter "a". Yet, only about a half of them relate it to the vertex location. Regarding parameter "c", whose function is similar to that of parameter "b" in the linear equation - 60% responded correctly. Merely a third responded correctly with regard to the relatively complicated role of parameter "b" in the parabola. This parameter has not specific role, being dependent on the symbols and the value of the other two parameters "a" and "c", which makes understanding its role more difficult.

It would appear that the examinees understand the functions of the parameters of the linear functions better than those of the quadratic functions. This difference is particular striking when comparing the roles of the parameter "b" in the direction equation $y=ax+b$ to parameter "c" in the parabola equation $y=ax^2+bx+c$. To both parameters a similar role in determining the intersection with axis y. Despite this the percentage of correct answers concerning the role of the parameter in the linear function is noticeably higher than the results received concerning the parameter in the quadratic function.

# 'TangentField': A tool for 'webbing' the learning of differential equations

*Margaret James, Phillip Kent and Phil Ramsden*
*Mathematics Department, Imperial College, University of London*

In *Mathematica*, a computer algebra system, we have programmed a tool (`TangentField`), which produces 'tangent fields' for first-order ordinary differential equations (ODEs)—loosely, pictures of the plane peppered with tangent stubs. Someone who already knew about ODEs would be able to see a tangent field as a visual representation linking a first order differential equation and its solutions. For the learner, we propose quite a different role for this tool—as a *starting point* for the development of a concept image for differential equations (Tall and Vinner 1981). Noss and Hoyles (1996) have extensively researched the idea of constructing computational tools in which the expert perceives an embedded structure but which are also intended to be, in Papert's sense, *syntonic* with the student's developing conceptions. They coined the term *webbing* for such a structured, yet locally responsive, learning environment.

We do not wish to use `TangentField` as a way of replacing the formal meaning of differential equation: on the contrary, we intend that entry into the formal discourse should be facilitated by these computational activities. For that to happen, the student must come to recognise explicitly the general structure that they are relying on implicitly in their work with `TangentField`. (In the meantime, they *can* rely on it, and build a rich conceptual image before formal definitions are needed). They must also forge links between the representation of general structure as expressed in this medium and those they meet in the formal discourse.

We present some of the activities that we devised with these aims in mind. We indicate how we have tried to instantiate the theoretical framework proposed by Noss and Hoyles at the microlevel of activities and accompanying texts. We present an analysis of some biographies of student meaning-making and say how we see this analysis feeding into the next cycle of design and theorizing.

## References

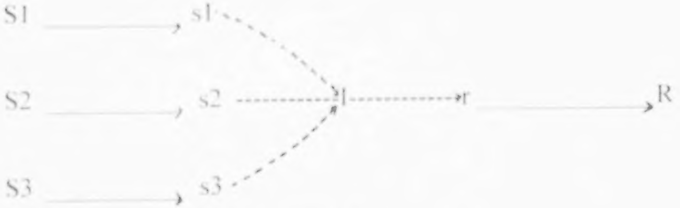
Tall, D. and Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

Noss, R. and Hoyles, C. (1996). *Windows on Mathematical Meanings*. Dordrecht: Kluwer Academic.

# STIMULUS COMBINATION AND MULTIPLE CAUSATION IN CHILDREN'S AREA JUDGEMENTS

Gillian Kidman
Centre for Mathematics and Science Education,
Queensland University of Technology, Kelvin Grove, Australia

Information Integration Theory (IIT) placescentral emphasis on problems of stimulus combination and multiple causation. The majority of judgements and actions reflect the combined influence of multiple stimuli. Anderson (1980) illustrates the nature of integration theory in terms of the functional measurement diagram of Figure 1.



(Modified from Source: Anderson, 1980, p.2)

All physical stimuli (S) impinge on a person and are converted by a valuation function to become psychological representations (s). These psychological stimuli are combined by the integration function (I) into an implicit response (r). Lastly, this implicit response is transformed by an output function to become an observed response (R). Thus, there are three operations that lead from the observable stimulus to the observable response:

(1) The valuation function relates physical intensity to psychological sensation
(2) The integration function (I) is perhaps the most important operation. It relates stimuli and response at the psychological level.
(3) The response function negotiates between the implicit response and the arbitrary constraints of the measuring device employed by the investigator.

Operations 1 and 3 above, are associated with problems of stimulus and response measurement. It is the integration function (I) that has the fundamental role. This poster will show 3 case studies concerningchildren's area judgements and integration rules.

Reference: Anderson, N.H. (1980). Information integration in developmental psychology. In F. Wilkening, J. Becker, & T. Trabasso (Eds.), Information integration by children (pp. 1-45). Hillsdale, New Jersey: Lawrence Erlbaum.

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# STUDENT TEACHERS' PICTURES OF TEACHING AND LEARNING

Signe Holm Knudtzon
The University of Bergen and Vestfold College, Norway

How is student teachers' understanding of mathematics education and children's learning in mathematics? Do they envision their teaching to be dominated by mathema-tical rules introduced by themselves with the children working individually in a silent classroom? Or do they wish to create a situation in which the children themselves get a chance to explore and discuss? When the students teach, what do they do, and what do they think they do?

The «picture» of mathematics and mathematics education that many student teachers have as a starting point seem to have the following components:
- mathematics is a subject with answers which are either right or wrong
- pupils tend to either like or dislike mathematics
- mathematics teaching consists of a teacher who explains how to do different tasks and pupils who follow the given procedures, doing lots of exercises.

For most of the students, their picture of mathematics and mathematics education seems to become more nuanced throughout their teacher's education, and the students present thoughts about co-operation, the use of language, problem-solving and working with concretes.

As a theoretical background for my analysis I use constructivism (Jaworski, Lerman, von Glaserfeld (radical-) and social constructivism (Ernest) and Activity Theory (Vygotsky, Mellin-Olsen, Linden). My project (Knudtzon) is a part of a PhD program under the Department of Psy-chology in The University of Bergen, Institute of Practical Pedagogy (IPP).

References:

Ernest, P. (1991) *The Philosophy of Mathematics Education*. London. Falmer Press.
Jaworski, B. (1994) *Investigating Mathematics Teaching*. A Constructivist Enquiry. London. Falmer Press.
Jaworski, B. (1996) A study of teacher enquiry into the processes of mathematics teaching. In Puig, L. and A. Gutiérrez (Eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of mathematics Education*. Vol.3, pp 129-136. Valencia. Spain.
Knudtzon, S. H. (1996) Student teachers' understanding of mathematics education. In Puig, L. and A. Gutiérrez (Eds.) *Proceedings of the 20th Conference of the International Group for the Psycho-logy of mathematics Education*. Vol.1 p181 Valencia. Spain.
Lerman, S. (1989) 'Constructivism, mathematics and mathematics education', *Educational studies in Mathematics*, 20, Kluwer, The Netherlands, pp. 211-33.
Linden, N. (1989) *Stilaser om barns læring*. Rådal. Caspar forlag.
Mellin-Olsen, S. (1991) *Hvordan tenker lærere om matematikkundervisning?* Landås. Bergen lærerhøgskole
Perry, W. G. (1970) *Forms of Intellectual an Ethical Development In the CollegeYears: A Scheme*. New York. Holt, Rhinehart & Winston.
von Glaserfeld, E. (1983) 'Learning as a Constructive Activity', in PME-NA Proceedings, Montréal. September - October, pp 41-69.
Vygotsky, L.S. (1979) *Mind in Society*, London, Harvard University Press.

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# DIFFICULT MATHS TOPICS TO TEACH:
# SOME STUDENT TEACHERS' VIEWS

Jean Melrose
Education Department, Loughborough University, Loughborough, U.K.

A group of students who are preparing to become secondary school (pupils aged 11-18) mathematics teachers had completed their first main teaching practice and were about to start on their second and final teaching practice. They were asked,

*"What are the topics in Maths that you found most difficult to teach?"*

Their replies were:-

- The things you don't think need explaining but they do.
- Fractions, because they have been taught it before in different ways.
- Topics that *are* difficult like negative numbers or starting algebra.
- Three dimensional work - sometimes they can't see what is meant.
- Some pupils get area and perimeter mixed up.

There was discussion about the variation between different groups of pupils, that some individuals, groups or classes seemed easier to teach than others. Other factors ranging from the time of day, to the lesson the pupils had come from, to the weather, were cited as relevant. Interestingly, no student mentioned the influence his or her own attitudes might have on the teaching.

More detailed inquiry into this meeting of the 'learning of mathematics cultures' of pupils and student teachers is underway.

References:

| | |
|---|---|
| ed. Hart K.M.(1981) | *Children's Understanding of Mathematics: 11-16*, John Murray, London |
| Bishop A.J.(1988) | *Mathematical enculturation*, Kluwer, Dordrecht |
| Sierpinska A.(1994) | *Understanding in Mathematics*, Falmer, London |

# COMPARATIVE STUDY ON PUPILS' PROGRESS IN MATHEMATICS (KASSEL PROJECT): ON THE RESEARCH RESULTS IN JAPAN

Tadao Nakahara, Masataka Koyama
Hiroshima University, Japan

This study is based on Kassel Project that aims to carry out research into the teaching and learning of mathematics in different countries, and ultimately to make recommendations about good practice in helping pupils achieve their mathematical potential. It is a main characteristic that the study is a longitudinal for two years and investigates mathematical potential as well as mathematical content-attainment of pupils in lower secondary schools.

In Japan, we chosen 1145 second-graders (13+) at lower secondary school level from six different public schools in taking their locality into account. Those pupils were administered the potential test and the number test in 1995 for the first year (13+) and the same number test in 1996 for the second year (14+). The potential test has been designed to assess the mathematical potential of pupils, and is marked out of 26 and lasts 40 minutes. The number test is one of four topic tests that will provide a measure of attainment in the number area, and is marked out of 50 and lasts 40 minutes.

Using the complete data of pupils (1093) involved in both tests of potential and number in 1995, the mean result of the potential test was obtained, and it is shown in Table 1 with reference to those of England and Germany. We show the mean result and progress of the number test in Table 2 with reference to those of England and Germany, by analyzing the sample data of pupils (198) administered the number test in both 1995 and 1996.

Table 1. Mean result of the potential test (Score out of 26)

| Japan | England | Germany |
|---|---|---|
| 14.8 | 12.5 | 13.7 |

Table 2. Mean result of the number test (Score out of 50) and progress for two years

| | First Year (13+) | Second Year (14+) | Progress |
|---|---|---|---|
| Japan | 26.9 | 30.0 | 3.1 |
| England | 17.6 | 20.2 | 2.6 |
| Germany | 23.5 | 26.9 | 3.4 |

In summary, these results show that pupils chosen in Japan performed better in both tests of potential and number than either England or Germany. The poster presentation will show more detail data and sample items, discuss some factors which lead to the difference, and suggest implications for improving mathematics teaching and learning.

***Reference***:

Burghes, D. and Blum, W., 1995, The Exeter-Kassel Comparative Project: A Review of Year 1 and Year 2 Results. ***Proceedings of a Seminar on Mathematics Education***, The Gatsby Charitable Foundation, 13-28.

# SPATIAL THINKING AND AREA CONCEPTS

**K. Owens, Uni. of Western Sydney & L. Outhred, Macquarie Uni.**

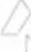
Students' ability to think spatially (e.g., mentally change the position of shapes and recognise parts of shapes) is likely to affect students' approaches to tasks involving estimating of area. It has also been suggested that students who are more advanced on spatial thinking will visualise and not draw. In this study 170 students aged 7 to 10 attempted the test *Thinking about 2D Shapes*. One set of items asked students whether a given tile (rectangular or triangular) could be used to cover a given figure. It was found that about half the students had difficulties, especially with giving the number of tiles required.

In a multivariate analysis the grouping variable of drawing/not drawing for all of the five items did not significantly explain variance in the scores on the test; better students visualised solutions and did not draw. The analysis of drawings suggested that this was due to drawing difficulties.

Overall spatial thinking, as measured by total scores on the test *Thinking about 2D Shapes*, is correlated with responses to the tiling items. For four items correct responses for the number of tiles needed was associated with higher overall scores for 2D spatial thinking (see Table 1). Distance from the tile and order of items might account for differences in the results for the two forms. Responses for Item 7 were not significantly related to the scores on the test but students' drawings showed that it was difficult for them to visualise or draw the correct-sized rectangle for tiling the square. A Rasch analysis had shown these items to fit an underlying trait because allowance was made for partially correct responses (Owens, 1992).

Table 1

F ratios for Relationships between Total Scores on Test *Thinking about 2D Shapes* & Giving Correct/Incorrect Number of Tiles

| Item | | | F ratio | | Significance of F | |
|---|---|---|---|---|---|---|
| | Tile & | Figure | Form H | Form S | Form H | Form S |
| 2 | | | 14.05 | 15.83 | .000** | .000** |
| 3 | | | 7.15 | 11.76 | .009** | .001** |
| 4 | |  | 3.31 | 7.33 | .072 | .008** |
| 5 | | | 3.46 | 23.16 | .066 | .000** |
| 7 | | | .32 | .38 | .473 | .537 |

*Note.* 92 students attempted Form H and 89 students attempted Form S.

** The grouping variable *correct/incorrect* for the item was very significantly related to overall score, $p < 0.01$ in the multivariate analysis of variance. The other grouping variable *drawing/not drawing* was entered simultaneously but was not significant for any item.

***Reference:*** Owens, K. (1992). Spatial mathematics: A group test for primary school students. In M. Stephens & J. Izard (Eds.), *Reshaping assessment practices: Assessment in the mathematical sciences under challenge* (pp. 333-354). Melbourne: ACER.

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# CONTRIBUTION OF SATD (DATA ACQUISITION AND TREATMENT SYSTEMS) IN GRAPHICS ANALYSIS AND INTERPRETATION

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Graphics are an information source in all school books, software and in pupils science tasks (Peixoto, 1996).

In scientific learning graphics are kee symbols and the main way of communication with students. Through it physical phenomena are analysed in function of time, and results and variables related to that phenomena are established. ...The inadequate knowledge of the skills to do graphics is pointed, by many researchers, as the main obstacle to the understanding of scientific concepts (Sham, Padilla and Mackenzie, 1986, in Mokros, 1987).

Through graphic form and with the utilisation of SATD the student makes a direct link with the real time the experiment is done and the symbolic representation of that experiment.

This system is extremely informative, because it presents all data in a visual way, and can be observed at the same time by a large number of students. The students can select points they want to do by hand and make a zoom of the scales for the experimental exploration of data (Rogers, 1987). It can still be used as a data library to compare different physical phenomena.

With this system we also can establish mathematical relations between different variables, do graphics representation of those relations, and do the integration and differentiation in a given time.

Taken this into account a study was done with 38 Maths and Science pre-service teachers, that as students had studied different concepts involving graphics analysis using traditional methods.

In this study the opinions about the possibility of using the system in their classes as well about the advantages of using it were analysed. The teachers conceptions before and after the utilisation of this system in simple experiments were analysed and the changes in the understanding of the concepts by the teachers were verified.

The results pointed out that the utilisation o the SATD leads to a better learning through the development of critical attitudes towards data acquisition and results discussion.

References

Peixoto, A. M. (1996). *Integração do Sistema de Aquisição e Tratamento de dados no Trabalho Experimental e na Formação de Professores de Ciências.* Tese de Mestrado. Aveiro: Universidade de Aveiro.

Mokros, J. R. e Tinker, R. F. (1987). The Impact of Microcomputer-Based Labs on Children's Ability to Interpret Graphs.*Journal of Research in Science Teaching*, vol. 24, nº 4, pp. 369-383.

Rogers, L. T. (1987). The computer-assisted laboratory. *Physics Education*, vol. 29, nº 1, pp. 46-50.

# NEW TRENDS IN RUSSIAN MATHEMATICS EDUCATION: IMPLICATIONS FOR TEACHER EDUCATION

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Since 1950-s, almost all psychology-based research in Soviet mathematics education has been developed under the strong influence of ideas of L.S.Vygotsky. Contrary to J.Piaget who supposed that teaching process should follow the psychical development of a child, L.S.Vygotsky proposed the idea that "pedagogy should aim at the tomorrow's, not yesterday's state of the child's development". The idea of "developing instruction" has been dominant in Soviet educational research during the post-war period.

At the present time we have several relatively new trends in mathematics education:

1)Didactical system of L.Zankov (for primary school);

2)Theory of "developing instruction" of V.Davydov (mainly for primary school, too);

3)Theory of "controlled process of learning" (N.Talyzina);

4)Differentiation and revision of the content of mathematics instruction (V.Gusev, G.Dorofeev);

5)"Pragmatic technology of mathematics teaching" (M.Volovich, G.Levitas).

These theories have found sporadic applications in courses organized within the system of in-service teacher training. However, almost no effect of these doctrines on regular teacher education is seen so far. We will outline some possible ways of application of mathematics-educational research in teacher training.

First, it is necessary to include elements of new theories into mathematics education courses in pedagogical institutes, to show various didactical approaches to students.

Second, it would be interesting to try to explicitly apply new theories of mathematics teaching in standard mathematical courses at universities and pedagogical institutes. Third, very promising seems the idea to implicitly intertwine didactical component with mathematical courses. This idea was suggested by several authors (S.Lerman, E.Ch. Wittman, H.C Reichel and earlier by G.Polya and A.Mordkovich).

Speaking about higher mathematics teaching, one should not forget that here mere correct, complete and organized presenting of the subject does not suffice. The style of conducting lectures and seminars for teaching at the undergraduate level is extremely important.

# Can IT-based Reengineering make Mathematics a Human Right?

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**Goal.** At the IT-college Denmark observations have shown, that many students show a poor math performance, a dislike of math and negative attitude towards their own learning potential in math. The college has asked for IT-based reengineering of Mathematics to make it a tool for all students.

The **ideology** of this research program is "Numeracy is a human right". To live as a human being in a society you need literacy and numeracy in order to communicate about qualities and quantities. Thus you need two languages. A "word language" that assigns words to things by means of sentences. And a "number language" to assign numbers or calculations to things by means of equations.

**Students** are considered belonging to a generation of post modernism in the sense of Giddens and Ziehe. The authority of the school will be accepted only if authentic situations can legitimise the concepts treated.

**Mathematics** is considered invented by man for a purpose in a given cultural and technological context and thus changing with these factors. Thus the appearence of IT-technology calls for reengineering mathematics differentiating it into concrete mathematics and abstract mathematics. Concrete mathematics is defined in a nominalistic or constructivist tradition based upon the theory of Vygotsky and Luhmann to be names, that are created to give man the ability to assign numbers to real world objects through measuring or calculations. Mathematics thus become a "language house" with only two levels of abstraction: Language and meta language.

**Learning** according to Jean Lave has to be situated in contexts where it is relevant an necessary to be meaningful. Vygotsky talks about the development of concepts from the concrete level through the zone of proximal development to abstract levels. Replacing abstract context free mathematics with concrete context situated mathematics creates a basis for accommodating the students knowledge and changing attitudes from negative to positive.

**The action.** Being situated in a concrete context numbers are differentiated into absolute unit numbers and relative per numbers (m, m/s, m/m=%). Operations are considered techniques for uniting and separating unit numbers (+ and -, and /) and for uniting and separating per-numbers (^ and √ log, ∫ and d/dx) according to the meaning of algebra: reuniting. IT-technology can solve any equation. So formulating and evaluating equations become the main human tasks in the future.

**Methods.** The effect of concrete mathematics on the students' level of competence and on the students' attitudes towards mathematics and towards their own learning potential will be examined through tests and interviews. Also the attitudes of teachers and the ministery will be examined through interviews. The analysis of the data will be based upon narrative and motivational theories.

# PROGRAMMING LANGUAGE LOGO IN SCHOOL MATHEMATICS AND TEACHER-TRAINING

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The programming language LOGO has several good features for being an excellent tool for studying and teaching mathematics. Simple syntax and good access to graphics offer good possibilities to study the mistakes one has made and contemplate their reasons and consequences. LOGO lets us study our own thinking process. Seymour Papert, the author of LOGO, calls attention to the so-called Mathland's concept that allows dealing with several mathematical problems from inside, so to say. LOGO can be used both for teaching relatively young school-children and also for pre-service and in-service teacher-training.

The first apprentice course in LOGO for the Estonian teachers took place in 1996 from January to April via e-mail. Another such course was held from November 1996 to March 1997. The courses were meant for teachers of different subjects. As it turned out, a large percentage (ca 45 per cent) of the participants were the teachers of mathematics. As the course was also organized by mathematicians, several problems encountered in school mathematics were dealt with. In addition to geometrical problems (that can be studied quite well using turtle geometry), other fields of mathematics (e.g. algebra, combinatorics, etc.) can be studied also.

It is interesting to mention that a large part of the participants came from the country (Tallinn, the capital of the republic, having the population of approximately one third of the total population of Estonia, was totally unrepresented). This shows that Internet gives us the possibilities to reduce the differences in training due to geographical location.

In future we plan to arrange an international course in LOGO (with English as the study language) for teachers of mathematics.

## References

Papert, S. 1980. Mindstorms: Children, Computers, and Powerful Ideas. New York: Basic Books.

Tõnisson, E. 1996. LOGO süsteem ja selle kasutamine koolimatemaatikas. (Diss.) Tartu.

# CONCEPT-IMAGES OF THE DEFINITE INTEGRAL IDENTIFIED IN SECONDARY SCHOOL STUDENTS (14-16 YEARS)

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In this Poster I present the development of my doctoral dissertation (Turégano, 1994).

Taking into account the historical genesis of the concept of integral (Turégano, 1993), I propose a theoretical model (Turégano, 1991, 1992, 1994) based on Lebesgue's studies (1928, 1931) on magnitudes and integral, which I have used in the making of a didactical proposal for introducing the concept of definite integral via its geometrical definition to first year secondary students who have not yet been initiated in the study of infinitesimal calculus.

My hipothesis is that students can learn (intuitively) calculus concepts without the previous or simultaneous command of the usual algorithmic skills, by using visualization on the computer to make sense of the concept of definite integral and its proprieties through the idea of area under a curve.

The initial sample was composed of three groups of first year students from three secondary schools in Albacete (Spain). This experience was carried out during the 1991-92 school year with a total of 87 students.

Firstly, through a questionnaire followed by a clinic interview, I wanted to know the students' previous ideas and first intuitions about infinity and limits, their skills for visualising and the concept-images of area they use in solving area problems, three issues that, from my point of view, may influence the formation of the concept of integral. The conclusions obtained, along with the theoretical model, allowed me to design didactic situations on the basis of the constructivist model of learning.

The experimental phase took place in May 1992 through 8 sessions during which the designed material was worked on. In four of them we used the computer, an indispensable tool for this kind of approach to the concept of integral.

After the learning stage I interviewed some students (selected through factor analisis of correspondences) while they were solving problems concerning the area under a curve in three different contexts mathematical, area under the curve speed and geometrical probabilities in order to determine the concept-images of definite integral formed by the students and how they carry out the transfer to contexts other than the mathematical one.

References

LEBESGUE, H. (1928). *Leçons sur l'intégration et la recherche de fonctions primitives.* Paris: Éditions Jacques Gabay. (Réimpression Gauthier-Villars et C$^{ie}$ Éditeurs, 1989)

-1931. *Measure and the integral.* Berkeley: Holden-Bay (Reprint, 1966)

TURÉGANO, P. (1991). *Propuesta didáctica para la integral definida.* IV JAEM, Castellón, 20-23 mayo, 1991.

-1992. Una alternativa a la integral Riemann. *Ensayos*, 6, 227-233.

-1993. *De la noción de área a su definición.* Cuenca: Servicio de Publicaciones de la Universidad de Castilla-La Mancha.

-1994. *Los conceptos en torno a la medida y el aprendizaje del cálculo infinitesimal.* Tesis doctoral en microfitxes. Servei de Publicacions de la Universitat de València.

# KNOWLEDGES OF PRESERVICE TEACHERS: A CLOSER LOOK THROUGH TWO MATHEMATICAL PROBLEM SOLVING ACTIVITIES

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Current recommendations for a change in mathematics education necessitate the reform of teacher preparation programmes. This mathematical preparation of teachers must enable them to, among others things, organise bodies of knowledge. Not only the knowledge that has to be taught, but also the knowledge of teaching (Schulman,1985). On the other hand as problem solving approaches to teaching mathematics is becoming in last decades important, it became clear that successful implementation depend upon the ability of teachers to incorporate such approach into their programmes (Taplin, 1996). So, research efforts must be direct to provide teachers with classroom ideas for translating those recommendations into classroom practice.

The underlying idea of this exploratory research has to do with the need to get a better understanding of preservice teachers' knowledges through some problem solving activities. So, this research involved a class of the 4th year of the Maths and Science course in a School of Higher Education during last year. We studied in this class the performances and knowledge of the students, future teachers, when they were involved in problem solving activities. In particular we want to know how they think to use calculator in mathematical instruction.

The following are the fundamental questions: (1) What level of performance revealed the participants in the written problem solving activities?; (2) What mathematical content knowledge and pedagogical knowledge revealed the participants in the written problem solving activities?; (3) What pedagogical importance the participants give to the calculator in the performance of these activities? and (4) The co-operative work did enrich the performance level and knowledges of the participants?

Taking into account the nature of the overall purpose of this study, we decided to focus on a qualitative methodology. In the process of data collection were used two activities and two openned-questionnaires. The activities proposed were from the NCTM Curriculum and Evaluation Standards and unifies and interrelates several mathematical themes and suggest the use of a simple calculator. The activities and questionnaires were done individually, but one of them was done in co-operative work too. The data analysis was holistic, descriptive and interpretative.

This presentation will try to answer the questions above and will consider to draw some conclusions.

# PRESERVICE TEACHERS' IDEAS OF CLASSROOM INVESTIGATIONS IN A COMPUTER ENVIRONMENT

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The NCTM Standards (1989) maintain that the appropriate integration of technology will "transform the mathematics classroom into a laboratory ... where students use technology to investigate, conjecture, and verify their findings ... [and] the teacher encourages experimentation" (p. 128). In this context, teachers often become students as they struggle with understanding technology and its pedagogical implications (Norman, 1993).

We teach prospective middle grades and high school teachers an undergraduate course devoted exclusively to using technology to teach mathematics. Over the course of several years we observed that finding corresponding strategies and then deciding on the types of questions that can be answered through the use of the technology tool appears to be very difficult and a major obstacle for many preservice teachers as they attempt to integrate technology into their plans for instruction. Therefore, this study focuses on preservice teachers' ideas and choices of problem-solving situations and strategies, as well as, their decisions on how technology can be used to help middle grades and high school students construct their knowledge of the mathematical concept under consideration.

Our data analysis show that some preservice teachers see the computer as a drawing or computational tool to enhance their lectures or class presentations. Others envision a classroom where technology is used for students to investigate mathematical ideas. However, this preliminary study finds preservice teachers' ideas about investigations are qualitatively different. The simple investigations for students tended to emphasize constructions and measurements using the GSP. Less common among preservice teachers' ideas of investigations was the notion that each construction and the related measure(s) were tied to an instance of reasoning and/or mathematical processes.

## References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers' of Mathematics.

Norman, F. A. (1993). Integrating research on teachers' knowledge of functions and their graphs. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representations of functions*. Hillsdale, NJ: Lawrence Erlbaum.

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